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Weibull analysis for normal/accelerated and fatigue random vibration test

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Abstract

In this paper, the formula to estimate the sample size n to perform a random vibration test is derived only from the desired reliability ($R(t)$). Then, the addressed n value is used to design the ISO16750-3 random vibration test IV for both normal and accelerated conditions. For the normal case, the applied random vibration stress (S) is modeled by using the Weibull stress distribution [$W(s)$]. Similarly, for the testing time (t), the Weibull time distribution [$W(t)$] is used to model its random behavior. For the accelerated case, by using the over-stress factor fitted from the $W(t)$ and $W(s)$ distributions, four accelerated scenarios are formulated with their corresponding testing's profiles. Additionally, from the $W(s)$ analysis, the stress formulation to perform the fatigue and Mohr stress analysis is given. Since the given Weibull/fatigue formulation is general, then the formulas to determine the $W(s)$ parameters, which correspond to any principal stresses values and/or vice versa, are given. Although the application is performed to demonstrate $R(t) = 0.97$ by testing only $n_2 = 6$ parts, the guidelines to use the values given in columns n, S, and t of the Weibull analysis table to generate several accelerated testing plans are given.

KEYWORDS

accelerated life test, fatigue analysis, ISO16750-3, reliability vibration test, sample size, Weibull distribution

1 | INTRODUCTION

Currently, a vibration testing is the general formal procedure applied to any mechanical component attached to a vehicle to confirm endurance and reliability ($R(t)$), as well as to provide manufacturers with dynamic information for structural analysis.[1] In a vibration test, the magnitude of the damage caused by the effect of the applied vibration on the component is directly related to the vibration amplitude. Because the vibration amplitude occurs at different frequencies, then in a vibration test, the expected (or observed) frequencies and the vibration magnitude are both given in a testing profile (here, the testing profile for ISO16750-3 norm test IV[2] is used). Therefore, since in the vibration test the testing profile is applied cyclically, the component's expected failure is, then, a fatigue failure.[3]

However, because no failures are allowed in the actual $R(t)$ vibration test, then no failure data are available to conduct the corresponding fatigue analysis (see section 2.5.1, page 49, in Castillo and Fernandez-Canteli[4]). Moreover, although the applied vibration is random and generates fatigue, the standard vibration test consists on demonstrating that the tested part presents a reliability of $R(t) = 0.97$ by testing without failure[5,6] $n = 23$ parts at the vibration stresses level (S) and testing time (t) given in the used testing profile. Hence, since no failures are allowed, if none of the tested parts fail after the test, we can conclude that the part presents at least the desired reliability of $R(t) = 0.97$. Unfortunately, because this vibration standard test neither relates $n = 23$ parts with $R(t) = 0.97$ nor offers failure time data to perform the probabilistic behavior of the applied vibration stress and its corresponding fatigue analysis, then the application of this standard test is not as efficient as it could be.

This paper responds to the fact that by applying the standard vibration test, we can perform neither the probabilistic analysis of the used profile nor the fatigue analysis that the applied vibration generates on the tested part. It does so based on the mechanical Weibull/stress formulation given in Piña-Monarrez[7] and considering that (a) the first row of the used testing profile represents the most severe damage generated by its vibration level, (b) the damage generated by the other testing profile rows can be added to the first one, and (c) the testing profile by itself represents one vibration cycle, which must be cyclically applied during the reliability test (see Section 4.2.1). The Weibull distribution as proposed in Piña-Monarrez[7] is used for both to represent the random behavior of the applied vibration profile and to derive the corresponding midrange (S_m) and alternating (S_a) stresses values[8] used to perform the corresponding fatigue analysis[9] (see Section 4.3).

Hence, since to apply the proposed Weibull/vibration/fatigue method, the sample size n, which completely represents the $R(t)$ index has to be first determined, then in Section 3.2 of this paper, based on the relation between the cumulative risk function $H(t)$ and the Weibull reliability function $R(t)$ given[10] as follows:

$$R(t) = \exp\{-H(t)\}. \tag{1}$$

The unknown n value depending only on $R(t)$ is determined, and because $H(t)$ in terms of the testing time t and on the Weibull shape (β) and scale (η) parameters is given as

$$H(t) = \left(\frac{t}{\eta}\right)^{\beta}. \tag{2}$$

Then, the derived n value for known β value completely represents the Weibull scale η parameter also. (see Section 4.2.3).

Hence, in the analysis using n, both the Weibull stress W(β, η_s) distribution used to model the random behavior of the profile used and the Weibull time W(β, η_t) distribution used to model the random behavior of the testing time are determined. Then based on both Weibull families, the analysis in Sections 4.2.1 to 4.2.6 is generalized to determine the vibrations testing parameters t_i, n_i, and S_i for different normal and accelerated testing's scenarios (see Table 3).

Finally, in the numerical application, vibration data given in appendix D of the user guide [11] for norm GMW3172 and the testing profile for the ISO 16750-3 test IV are used. The objective of the test is to demonstrate that a sprung mass product meets $R(t) = 0.97$. Since the derived n value lets us determine any desired Weibull scale η parameter, the test design and analysis are given for the normal, the accelerated, and the fatigue scenarios. It is important to mention that in the fatigue analysis, the fatigue material exponent is incorporated as an exponent in the over stress factor used to determine the vibration level, which should be applied during the test.

The paper is structured as follows: Section 2 presents the vibration testing's generalities. In Section 3, the expected Weibull scale η value, which n should represent, is given (Section 3.1), and the formula to estimate n in such a way that its value always represents η, is derived depending only on $R(t)$ (Section 3.2). Then, Section 4 presents the formulation of the proposed method to design and apply a random vibration test for all three conditions, normal, accelerated, and fatigue, as follows. In Section 4 the vibration data are given, and in Section 4.1, the actual standard vibration test is performed. In Section 4.2.1, the vibration Weibull stress β and η_s parameters are estimated directly from the testing profile used, while in Section 4.2.2, the corresponding Weibull testing time β and η_t parameters are determined based on the addressed Weibull stress parameters. In Section 4.2.3, the numerical Weibull stress and Weibull testing time analysis are presented. However, because in this initial Weibull analysis, the sets ($R(t)$ and n), and (S and t) fall in a different row of the Weibull analysis table (See Table 2), Section 4.2.4 offers the steps to set ($R(t)$ and n), and (S and t) in the same row. The numerical analysis is presented in Table 3 of Weibull testing's plan application section. Additionally, based on the row in Table 3 containing all the testing factors, Section 4.2.5 shows the accelerated testing design to test lower time t at an accelerated vibration level S and vice versa, as well as to test lower parts n at lower/higher t or S values. In Section 4.2.6, the accelerated fatigue factor is derived and applied to design the fatigue/accelerated scenarios in Section 4.2.5, and Section 4.3 features the formulation to determine the principal, midrange, and alternating stresses values as well as the guidelines to perform the Weibull/fatigue, Weibull/Mohr circle, and Weibull/Goodman analyses. Finally, conclusions are presented in Section 5.

2 | VIBRATION TESTING GENERALITIES

In practice, two types of vibration testing are performed, the swept sine and the random vibration testing. The first one is recommended when the objective is to determine the resonant frequency (see [12] SAE J1455-2017 section 4.10.4.1, page 28), while the second one is used when the aim is to excite the product at different frequency ranges (see SAE J1455-2017 section 4.10.4.2, page 28). Of the two, the random vibration testing is the most realistic method.[13]

On the other hand, in a random vibration testing, the most critical variables are mounting location, vibration frequency f, applied vibration level S, and sample size n. The mounting location is generally classified as in the cab (or passenger compartment), on the engine, and on the chassis (or under hood), while the vibration frequency f and the applied vibration level S both depend on the product's location. Thus, in a random vibration test, f and S are both given in the testing profile of the standard used (ISO16750-3, SAE J1455-2017, or GMW3172). Since in the testing profile, each one of the frequencies represents a stationary process,[14,15] then the testing parameters of the testing profile will not change significantly over a given period. As a result, it can be modeled by a probability density function, as it is the case of the Weibull distribution.[16] This fact implies that because in a Weibull vibration analysis, $R(t)$, t, f, and S are all known, and β is derived from the vibration testing's profile (see Section 4.2.1 and Equation 21), then n must be estimated in such a way that its value completely represents η. Following is the formulation to determine n.

3 | DETERMINATION OF n

The formula to estimate the n value, which always represents $R(t)$ and η is based on the following two facts. (a) From Equations (1) and (2), given $R(t)$, t, and β, the corresponding η value is unique and (b) because from Equation (1), the relation between $H(t)$ and $R(t)$ is unique, then the addressed n value is unique, and it is also completely determined. Therefore, since n has to represent η, it is necessary to first determine the η value that n should represent.

3.1 | Estimation of η

This section aims to show that for known $R(t)$, t, and β values, the corresponding η value is unique and that it is completely determined. Seeing this let rewrite Equation (1) as follows:

$$H(t) = -\ln[R(t)] \tag{3}$$

and rewrite Equation (2) as follows:

$$\eta^{\beta} = \left(\frac{1}{H(t)}\right)t^{\beta}. \tag{4}$$

Thus, by substituting Equation (3) into Equation (4)

$$\eta^{\beta} = \left[\frac{1}{-\ln\{R(t)\}}\right]t^{\beta}. \tag{5}$$

Hence, from Equation (5), finally, we have that η is unique for the known $R(t)$, t, and β values, and that it is given as follows:

$$\eta = \frac{1}{[-\ln\{R(t)\}]^{1/\beta}}t. \tag{6}$$

Now based on Equations (4) and (6), the formula to estimate n in such a way that its value always represents $R(t)$ and η can be derived.

3.2 | Derivation of n

In this section, the objective consists on deriving n in such a way that its value completely represents η. And since from the demonstration test plan inputs, we always know $R(t)$, then n is determined based only on $R(t)$. Moreover, because all the n tested parts are independent and identically Weibull distributed, then the n parts have the same shape β parameter (implying the n parts presents the same failure mode). Hence, because once β is known, only η has to be determined, then n has to completely represent η. And because the Weibull distribution is the minimum value extreme distribution, then based on its weakest link principle, which states that the failure of a system is determined by its weakest element (see Castillo,[17] page 25), in this section, n is determined based on the facts that (a) η represents the $R(t) = 0.3678$ percentile, which from Equation (1) occurs when $H(t) = 1$; (b) in the demonstration test plan, all the n products are independent and identically Weibull distributed, and all of them are tested at the same time t (no failures are allowed); and (c) from the Weibull stable property[4] section 2.3.1.2, page 38, the minimum cumulative time function $F(t_{\min})$ of the set of the n tested parts is given as follows:

$$F(t_{\min}) = 1 - [1 - F(t)]^n. \tag{7a}$$

Thus, since $F(t) = 1 - R(t)$, then Equation (7a) in terms of $R(t)$ is given as follows:

$$1 - F(t_{\min}) = \exp\left\{-n\left(\frac{t}{\eta}\right)^{\beta}\right\} = \exp\left\{\frac{-nt^{\beta}}{\eta^{\beta}}\right\}, \tag{7b}$$

which implies that the cumulative risk function $H(t)$ for a set of n variables is given as follows:

$$H(t) = \left\{\frac{nt^{\beta}}{\eta^{\beta}}\right\}. \tag{7c}$$

Therefore, since we are estimating η, implying that $1 - F(t_{\min}) = R(t) = 0.3678$ or equivalently that $H(t) = 1$, then from Equation (7c), the relation between η and t for a set of n variables is always given as follows:

$$\eta^{\beta} = nt^{\beta}. \tag{8}$$

From Equation (8), notice that because nt^{β} represents the total tested time, which η has to represent, and that t is powered to the exponent β, then because in Weibull analysis, β represents the effective intensity (spread) on which the applied stress affects the lifetimes. Then n in terms of t, β, and η is given by:

$$n = \left(\frac{\eta^{\beta}}{t^{\beta}}\right), \tag{9}$$

or equivalently from Equation (7c), n in terms of $H(t)$ is given by:

$$n = \frac{1}{H(t)} = \left(\frac{\eta^{\beta}}{t^{\beta}}\right). \tag{10}$$

Thus, as in Equation (6), by substituting Equation (3) into Equation (10), n in terms of $R(t)$ is given by:

$$n = \frac{1}{[-\ln\{R(t)\}]}. \tag{11}$$

Finally, by comparing Equations (5), (6), and (11), we have that the n value given in Equation (11) completely represents both $R(t)$ and η. Therefore, since $R(t)$ is a reliability index, which represents no failures, then Equation (11) also represents the n parts, which have to be tested without failures in order to demonstrate the tested product meets with the desired $R(t)$ index. Moreover, notice from Equations (9) and (10) that because n represents the times that the continuous t variable has to be tested, then n is a continuous variable also, and that for known t and η values, n is unique. Finally, note that by substituting Equation (11) into Equation (6), η in terms of n is given by:

$$\eta = n^{1/\beta}t. \tag{12}$$

And because n completely represents η and $R(t)$, then $R(t)$ in terms of n is directly given as follows:

$$R(t) = \exp\left\{\frac{-1}{n}\right\}. \tag{13}$$

Now that from Equation (11), we have the right n value to perform a vibration test, let us show numerically the vibration testing design and its analysis by using data given in appendix D of the user guide of the norm GMW3172.

4 | RANDOM VIBRATION TESTING DESIGN

In this section, data given in appendix D, page 276, of the user guide for norm GMW3172 are used to show how Equation (11) works in the standard vibration test. Data are as follows: "The basic sprung-mass specification for a car requires the product to be tested at 2.76 Grms with a specified Power Spectral Density (PSD) for 8 hours in each axis. This set of test conditions represents a life of 100,000 miles for a car. The recommendation is to test the product for 8 hours in the 'X' direction and then for 8 hours in the 'Y' direction. The 'X' and 'Y' directions are defined as being in the plane of the circuit board. The 'Z' direction is to be tested last. The 'Z' direction is defined as being perpendicular to the plane of the circuit board. Vibration testing is performed with superimposed thermal cycling and vibration occurring simultaneously, as well as under thorough product monitoring" (for details of the thermal cycling testing see appendix G of the user guide for norm GMW3172, page 290).

4.1 | Standard testing design analysis

Since data correspond to sprung mass mounted on a car, then the random vibration test IV for the ISO 16750-3 section 4.1.2.4, page10, should be performed under the following conditions.

Step 1. From the ISO 16750-3 section 4.1.2.4.2, the testing time on each one of the X, Y, and Z axes is of 8 hours. Therefore, the total testing time for the analysis is $t = 24$ hours, and the applied vibration level is 27.1 m/s^2 or equivalently $S = (27.1/9.80665) = 2.76$ Grms.

Step 2. Table 7 of the ISO 16750-3, section 4.1.2.4.2 shows the values of the PSD in acceleration units [rms^2 = (m/s^2)^2/Hz], while the frequencies are shown in Hertz (Hz). Both are shown in Table 1 and Figure 1.

Step 3. The operating mode during the test is the 3.2 mode given[18] in the ISO16750-1, section 5.3, which is "systems/components are tested with electric operation and control in typical operating mode."

Step 4. The functional status classification is the functional status A given in the ISO16750-1, section 6.2, which is "All functions of the device/system performed as designed during and after the test."

Step 5. As shown in the ISO 16750-3 norm and in appendix D of the user guide for norm GMW3172, "the vibration test should be performed with superimposed thermal cycling occurring simultaneously with vibration, and the product fully monitored." However, when Thermal shock and Power Temperature Cycle are applied using the Coffin Manson model, the test can be performed following appendix G of the user guide for norm GMW3172, page 290. Otherwise, the ISO 16750-3, sec 4.1.1 should be applied.

Step 6. Appendix C of the user guide of the norm GMW3172, page 275, shows that the desired reliability to be demonstrated is $R(t) = 0.97$ by testing only $n = 23$ parts. However, since from Equation (11), the right n value is known to represent $R(t) = 0.97$, in order to demonstrate $R(t) = 0.97$, instead of testing $n = 23$ parts, the recommendation is to

TABLE 1 ISO16750-3, test IV: Random vibration profile

Freq, Hz	PSD (m/s^2)^2/Hz	dB	Oct	dB/Oct	Area	rms
10.00	30.0000	*	*	*	300.00	17.32
400.00	0.2000	−21.76	5.32	−4.09	614.00	24.78
1000.00	0.2000	0.00	1.32	0.00	734.00	**27.09**

* it only indicates it is an empty cell.

Bold number shown readers the total accumulated energy.

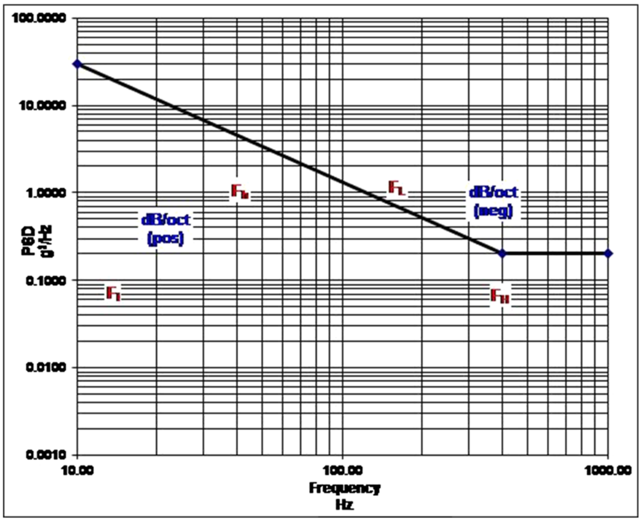

FIGURE 1 ISO16750-3, test IV testing profile [Colour figure can be viewed at wileyonlinelibrary.com]

test $n = 32.8308 \approx 33$ parts without failure for 24 hours each, at a constant vibration level of $S = 2.76$ Grms, and if none of them fail, it can be said that the tested product features the desired reliability of $R(t) = 0.97$ as a minimum.

As a summary of the standard vibration test, it can be said that because the standard vibration test requires to test only $n = 23$ parts (see appendix C of the user guide for norm GMW3172, page 278), then if only 23 parts were tested, the demonstrated reliability would be underestimated. In fact, from Equation (13), using $n = 23$, only $R(t) = \exp\{-1/23\} = 0.9574$ is demonstrated. Additionally, it should be noticed that because this standard test cannot be related to the Weibull distribution, then so far, it is not possible to numerically observe, from Equation (12) for example, that the sub-estimation of $R(t)$ occurred because $n = 23$ does not represent the η value.

Therefore, based on the used profile data given in Table 1, the next section explains the analysis to perform the corresponding Weibull probabilistic analysis.

4.2 | Probabilistic Weibull vibration analysis

First, it should be stated that besides the derivation of the right n value given in Equation (11), the main contribution of this paper lies this section. Such contribution consists of deriving the formulation and offering the guidelines to perform the probabilistic Weibull vibration analysis for both normal and accelerated conditions, as well as for the corresponding fatigue analysis. To facilitate its understanding, the section is presented in subsections. Section 4.2.1 presents the formulation to determine the Weibull stress W (β_s, η_s) distribution, which is used to model the random behavior of the applied vibration stress S, directly from the used vibration profile. Then based on the aforementioned W (β_s, η_s) distribution, Section 4.2.2 shows how the corresponding Weibull vibration testing time W (β_t, η_t) distribution is derived. Next, Table 2 in Section 4.2.3 shows the numerical application to data of Section 4 to show how both W (β_s, η_s) and W (β_t, η_t) can be used to model the random behavior of S and t. Because in this initial Weibull analysis, the sets ($R(t)$ and n) and (S and t) fall in a different row of Table 2, then in Section 4.2.4, the steps to perform the Weibull analysis on which ($R(t)$ and n) and (S and t) all fall in the same row are given. Weibull testing's plan application section features the numerical analysis, presented in Table 3, and based on the row in Table 3, which contains all the testing factors, in Section 4.2.5, the accelerated testing design for several scenarios is presented. Similarly, Section 4.2.6 presents the corresponding fatigue analysis to the accelerated scenarios of Section 4.2.5. Finally, in Section 4.3, the formulation to determine the principal, midrange, and alternating stresses values, as well as the guidelines to perform the Weibull/fatigue, the Weibull/Mohr circle, and the Weibull/Goodman analysis are given.

TABLE 2 Weibull/stress analysis to determine the minimum strength that the tested product would present to fulfills with $R(t) = 0.97$

Equation									
(11)	(20)	(23)	(24)	(25)	(11)	(26)		(27)	(30)
n	Y_i	$ln[tan(\theta_i)]$	$tan(\theta_i)$	$R(t_i)$	n_i	λ_{1i}	λ_{2i}	S_{Gi}	t_i
1	−3.85466	−1.54184	0.21399	0.9790	47.21	2192.91	100.41	4.78	8.03
	−3.49137	−1.39653	0.24746	**0.9700**	**32.83**	**1896.32**	116.12	**4.44**	9.29
2	−2.95192	−1.18075	0.30705	0.9491	19.14	1528.28	144.08	3.99	11.53
3	−2.47345	−0.98937	0.37181	0.9192	11.86	1262.07	174.47	3.62	13.96
	−2.30259	−0.92102	0.39811	0.9048	10.00	1178.70	186.82	3.50	14.95
4	−2.14209	−0.85682	0.42451	0.8892	8.52	1105.41	199.20	3.39	15.94
5	−1.88612	−0.75444	0.47028	0.8593	6.59	997.83	220.68	3.22	17.65
6	−1.67599	−0.67038	0.51151	0.8293	5.34	917.39	240.03	3.09	19.20
7	−1.49659	−0.59863	0.54957	0.7994	4.47	853.86	257.89	2.98	20.63
8	−1.33916	−0.53566	0.58528	0.7695	3.82	801.75	274.65	2.89	21.97
9	−1.19815	−0.47925	0.61925	0.7395	3.31	757.78	290.58	2.81	23.25
	−1.11842	−0.44736	0.63931	**0.7112**	**3.06**	**734.00**	**300.00**	**2.76**	**24.00**
10	−1.06979	−0.42791	0.65187	0.7096	2.91	719.86	305.89	2.74	24.47
11	−0.95142	−0.38056	0.68348	0.6796	2.59	686.57	320.72	2.67	25.66
12	−0.84108	−0.33643	0.71432	0.6497	2.32	656.93	335.20	2.61	26.82
13	−0.73726	−0.29490	0.74461	0.6198	2.09	630.21	349.41	2.56	27.95
14	−0.63878	−0.25551	0.77452	0.5898	1.89	605.86	363.45	2.51	29.08
15	−0.54467	−0.21787	0.80423	0.5599	1.72	583.48	377.39	2.46	30.19
16	−0.45414	−0.18165	0.83389	0.5299	1.57	562.73	391.31	2.46	31.30
17	−0.36651	−0.14660	0.86364	0.5000	1.44	143.35	405.27	2.38	32.42
18	−0.28118	−0.11247	0.89363	0.4701	1.32	525.11	419.34	2.34	33.55
19	−0.19759	−0.07903	0.92401	0.4401	1.22	507.85	433.60	2.30	34.69
20	−0.11523	−0.04609	0.95495	0.4102	1.12	491.39	448.12	2.26	35.85
21	−0.03360	−0.01344	0.98665	0.3802	1.03	475.60	462.99	2.22	37.04
22	0.04781	0.01912	1.01931	0.3503	0.95	460.37	478.31	2.19	38.27
23	0.12995	0.05182	1.05318	0.3204	0.88	445.56	494.21	2.15	39.54
24	021223	0.08489	1.08860	0.2904	0.81	431.06	510.83	2.12	40.87
25	0.29657	0.11863	1.12595	0.2605	0.74	416.76	528.36	2.08	42.27
26	0.38345	0.15338	1.16576	0.2305	0.68	402.53	547.04	2.05	43.76
27	0.47403	0.18961	120878	0.2006	0.62	388.21	567.22	2.01	45.38
28	0.56990	0.22796	1.25603	0.1707	0.57	373.60	589.40	1.97	47.15
29	0.67345	0.26938	1.30915	0.1407	0.51	358.44	614.32	1.93	49.15
30	0.78856	0.31542	1.37083	0.1108	0.45	342.31	643.27	1.89	51.46
31	0.92239	0.36895	1.44622	0.0808	0.34	324.47	678.64	1.84	54.29
32	1.09123	0.43648	1.54726	0.0509	0.34	303.28	726.06	1.78	58.08
33	1.35202	0.54080	1.71738	0.0210	0.26	2.73.24	805.89	1.69	64.47
$\mu y = -0.55479467$									
$\sigma y = 1.201931562$									

Bold shows readers the testing plan imputs (R(t), n) and (S, t) do not fall in the same row.

4.2.1 | Weibull/vibration stress parameter estimation

In this section, the objective is to determine the Weibull shape β and scale η parameters of the two parameter Weibull distribution, directly from the used vibration testing profile. The two parameter Weibull distribution is given by:

TABLE 3 Weibull testing's plan analysis

Equation							
(11) n	(20) Y_i	(23) $ln[tan(\theta_i)]$	(24) $tan(\theta_i)$	(25) $R(t_i)$	(11) n_i	(37) S_{Gi}	(39) t_i
1	−3.85466	−1.54184	0.21399	0.9790	47.21	2.39	20.75
	−3.49137	−1.39653	0.24746	**0.9700**	**32.83**	**2.76**	**24.00**
2	−2.95192	−1.18075	0.30705	0.9491	19.14	3.43	29.78
3	−2.47345	−0.98937	0.37181	0.9192	11.86	4.15	36.06
	−2.30259	−0.92102	0.39811	**0.9048**	**10.00**	**4.44**	**38.61**
4	−2.14209	−0.85682	0.42451	0.8892	8.52	4.74	41.17
5	−1.88612	−0.75444	0.47028	0.8593	6.59	5.25	45.61
6	−1.67599	−0.67038	0.51151	0.8293	5.34	5.71	49.61
7	−1.49659	−0.59863	0.54957	0.7994	4.47	6.14	53.30
8	−1.33916	−0.53566	0.58528	0.7695	3.82	6.53	56.77
9	−1.19815	−0.47925	0.61925	0.7395	3.31	6.91	60.06
10	−1.06979	−0.42791	0.65187	0.7096	2.91	7.28	63.22
11	−0.95142	−0.38056	0.68348	0.6796	2.59	7.63	66.29
12	−0.84108	−0.33643	0.71432	0.6497	2.32	7.97	69.28
13	−0.73726	−0.29490	0.74461	0.6198	2.09	8.31	72.22
14	−0.63878	−0.25551	0.77452	0.5898	1.89	8.65	75.12
15	−0.54467	−0.21787	0.80423	0.5599	1.72	8.98	78.00
16	−0.45414	−0.18165	0.83389	0.5299	1.57	9.31	80.88
17	−0.36651	−0.14660	0.86364	0.5000	1.44	9.64	83.76
18	−0.28118	−0.11247	0.89363	0.4701	1.32	9.98	86.67
19	−0.19759	−0.07903	0.92401	0.4401	1.22	10.32	89.62
20	−0.11523	−0.04609	0.95495	0.4102	1.12	10.66	92.62
21	−0.03360	−0.01344	0.98665	0.3802	1.03	11.02	95.69
22	0.04781	0.01912	1.01931	0.3503	0.95	11.38	98.86
23	0.12995	0.05182	1.05318	0.3204	0.88	11.76	102.15
24	021223	0.08489	1.08860	0.2904	0.81	12.15	105.58
25	0.29657	0.11863	1.12595	0.2605	0.74	12.57	109.20
26	0.38345	0.15338	1.16576	0.2305	0.68	13.01	113.06
27	0.47403	0.18961	120878	0.2006	0.62	13.50	117.24
28	0.56990	0.22796	1.25603	0.1707	0.57	14.02	121.82
29	0.67345	0.26938	1.30915	0.1407	0.51	14.62	126.97
30	0.78856	0.31542	1.37083	0.1108	0.45	15.30	132.95
31	0.92239	0.36895	1.44622	0.0808	0.340	16.15	140.26
32	1.09123	0.43648	1.54726	0.0509	0.34	17.27	150.06
33	1.35202	0.54080	1.71738	0.0210	0.26	19.17	166.56

Bold shows readers all the testing inputs (R(t),n) and (S,t) are now in the same row.

$$f(t)=\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\}, \tag{14}$$

and based on the vibration testing profile, their parameters are determined under the following three assumptions: (a)

The first row in the testing profile represents the test's most severe scenario, (b) the generated damage in one cycle is cumulated from the first row of the testing profile to the last, and (c) the whole test profile is considered a compact cycle, which is repeated *n* times until failure occurs. Based on these assumptions, the steps to determine the corresponding β and η parameters, directly from the testing profile are the following.

Step 1. From the first row of the testing profile, take the product of the applied frequency and energy as the minimum eigenvalue

$$\lambda_{\min} = f_1 {*} G_1. \tag{15}$$

Step 2. Take the maximum eigenvalue as the total cumulated energy of the testing profile as follows:

$$\lambda_{\max} = \sum_{i=1}^{k} A_i, \tag{16}$$

where Ai represents the area of the *th*-row of the testing profile, given as follows[19]:

$$A_i = 10\log(2)\frac{PSD_i}{10\log(2)+m}\left[f_i - f_{i-1}\left(\frac{f_{i-1}}{f_i}\right)^{m/10\log(2)}\right]. \tag{17a}$$

PSD_i is the applied energy and f_i is the frequency of the *i*th row of the used testing's profile, while $f_{(i\text{-}1)}$ is the frequency of the (th-1)-row of the testing's profile, and m is the slope given as follows:

$$m = dB/\text{octaves}, \tag{17b}$$

where

$$dB = 10\log(PSD_i/PSD_{i-1}), \tag{17c}$$

$$\text{octaves} = \frac{\log(f_i/f_{i-1})}{\log(2)}. \tag{17d}$$

Step 3. By using the maximum eigenvalue taken from step 2, determine the vibration stress level S in Grms units as follows:

$$S = \frac{\sqrt{\lambda_{\max}}}{9.80665} \tag{18}$$

Step 4. By using the n value estimated in step 6 of Section 4.1, determine the median rank approach[20] as follows:

$$F(t_i) = \frac{i-0.3}{\text{n}+0.4} \tag{19}$$

where $F(t_i) = 1 - R(t_i)$ is the cumulated failure time percentile.

Step 5. By using the $F(t_i)$ elements from Step 4, determine the corresponding Y_i elements as follows:

$$Y_i = \ln(-\ln(1 - F(t_i))) = bo + B\ln(t_i), \tag{20}$$

and from the Y_i elements, determine the corresponding arithmetic mean μ_y and the standard deviation σ_y.

Step 6. By using the μ_y value from step 5, and the addressed $\lambda_{\min}$ and $\lambda_{\max}$ values from steps 1 and 2, determine the corresponding Weibull vibration βs parameter[7] as follows:

$$\beta s = \frac{-4\mu_Y}{0.9947 {*} \ln(\lambda_{\max}/\lambda_{\min})} \tag{21}$$

Step 7. By using the addressed $\lambda_{\min}$ and $\lambda_{\max}$ values from steps 1 and 2, determine the corresponding Weibull vibration stress parameter as follows[7]:

$$\eta_s = \sqrt{\lambda_{\max}\lambda_{\min}} \tag{22}$$

The Weibull parameters addressed in Equations (21) and (22) are the parameters of the Weibull stress family W (βs, ηs), which completely represent the vibration profile of step 2 in Section 4.1. Now that by using Equations (15), (16), (21), and (22), the Weibull parameters can be determined directly from the vibration testing profile; the addressed W (βs, ηs) can be used to determine the corresponding Weibull testing time parameters.

4.2.2 | Weibull/vibration testing time parameters estimation

The objective of this section is to establish the Weibull testing time W (β_t, η_t) distribution, which determines the expected testing time values that correspond to each of the expected vibration stress levels S_i values given by the W (βs, ηs) distribution. The analysis to derive the corresponding W (β_t, η_t) distribution is performed by using both the Weibull stress βs and ηs parameters in Section 4.2.1, and the testing profile data $S = 2.71$ Grms and $t = 24$ hours shown in Table 1.

By doing this, (a) the expected S_i values are determined, (b) the scale testing time η_t parameter is determined, and finally by using both the addressed η_t value and the basic Weibull values, (c) the expected $t_{i,}$ value, which corresponds to each one of the S_i values, is determined. Each of these is determined as follows:

1. To determine the S_i values:

Step 1. Determine the desired $R(t)$ index, and from Equation (11), determine the corresponding n value.

Step 2. By using the n value in step 1, determine the Y_i elements, its mean $\mu_{y,}$ and its standard deviation $\sigma_{y,}$ as men tioned in step 5 of Section 4.2.

Step 3. By using the βs parameter from step 6 in Section 4.2, and the Y_i elements from step 2, determine the log arithm of the basic Weibull values as follows:

$$\ln[\tan(\theta_i)] = \left\{\frac{Y_i}{\beta_s}\right\} \tag{23}$$

Step 4. From the logarithm of the basic Weibull values in step 3, determine the corresponding *basic Weibull* values as follows:

$$\tan(\theta_i) = \exp\left\{\frac{Y_i}{\beta_s}\right\} \tag{24}$$

Step 5. By using the basic Weibull values in step 4, determine their corresponding reliability indices as follows:

$$R(t_i) = \exp\left\{-\left(\tan(\theta_i)^{\beta}\right)\right\} \tag{25}$$

Step 6. By using the reliability indices in step 5 in Equation (11), determine their corresponding n_i value, which corresponds to each one of the the basic Weibull values of step 4.

Step 7. By using the basic Weibull values in step 4 and the ηs parameter from step 7 in Section 4.2, determine the expected maximum and minimum vibration levels as follows:

$$\lambda_{\max(i)} = \lambda_{1i} = \eta s/\tan(\theta_i) \quad \text{and} \quad \lambda_{\min(i)} = \lambda_{2i} = \eta s^* \tan(\theta_i) \tag{26}$$

Step 8. From the maximum vibration levels in step 7, determine the corresponding vibration stress value S_i for each one of the basic Weibull values as follows:

$$S_i = \frac{\sqrt{\lambda_{\max(i)}}}{9.80665} \tag{27}$$

The S_i values in these steps represent the expected vibration stress values given by the Weibull stress W (βs, ηs) distribution.

2. The corresponding scale testing time η_t parameter is determined as follows:
   Step 9. By using the addressed $\lambda_{\min}$ and $\lambda_{\max}$ values from steps 1 and 2 in Section 4.2, determine the corresponding basic Weibull value as follows:

$$\tan(\theta_{\lambda s}) = \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}} \tag{28}$$

   Step 10. By using the basic Weibull value from step 9 and the testing time from step 1 in Section 4.1, determine the Weibull scale testing time parameter in time units as follows:

$$\eta_t = \frac{t}{\tan(\theta_{\lambda s})} \tag{29}$$

The βs parameter in Section 4.2.1 and the η_t parameter of this step represent the Weibull testing time family W ($\beta s = \beta t$, η_t), which in the Weibull analysis is used to determine the testing time t_i value that corresponds to each one of the S_i values.

3. The corresponding t_i elements are estimated as follows:
   Step 11. By using the basic Weibull values in step 4 and the Weibull testing time scale parameter from step 10, the testing time that corresponds to each one of the basic Weibull values is given as follows:

$$t_i = \eta_t{}^* \tan(\theta_i) \tag{30}$$

Next, the addressed stress Weibull family W (βs, η_S) in Section 4.2.1 and the Weibull testing time family W (βs, η_t) in this section will be used to present the numerical Weibull vibration application.

4.2.3 | Numerical Weibull stress and testing time analysis

In the application, the testing profile in Section 4.1 is used, thereby following the steps in Section 4.2.2,

1. The Weibull stress W (βs, η_S), as well as the expected vibration stress S_i elements, are estimated as follows.
   Step 1. $R(t) = 0.97$ and $n = 33$ parts.
   Step 2. The Yi elements, its mean μ_y, and its standard deviation σ_y are all shown in Table 2.
   Step 3. Since from Equation (21), with $\mu_y = -0.554795$ and $\lambda_{\min} = 300$ rms^2 and $\lambda_{\max} = 734$ rms^2, $\beta s = 2.5$, then by using its value in Equation (23), the logarithm of the basic Weibull values are shown in Table 2.
   Step 4. From Equation (24), the basic Weibull values are also shown in Table 2.
   Step 5. The reliability indices are featured in Table 2.
   Step 6. The n_i values are given in Table 2.
   Step 7. Since by using $\lambda_{\min} = 300$ rms^2 and $\lambda_{\max} = 734$ rms^2, in Equation (22), $\eta s = 469.2544$ rms^2, then by using this value in Equation (26), the expected maximum and minimum vibration levels are given in Table 2. Here, notice the Weibull stress distribution is W ($\beta s = 2.5$, $\eta_S = 469.2544$ rms^2).
   Step 8. The expected vibration stress S_i values are shown in Table 2.
2. The scale testing time parameter is derived as follows:
   Step 9. From Equation (28), the basic Weibull value that corresponds to $\lambda_{\min} = 300$ rms^2 and $\lambda_{\max} = 734$ rms^2 is $\tan(\theta_{\lambda s}) = 0.639312$.

Step 10. From Equation (29), the Weibull scale parameter in time units is $\eta_t = 37.54038$hours.
Step 11. From Equation (30), the expected testing times are given in Table 2. Here, it should be noticed that the Weibull testing time distribution is W ($\beta s = 2.5$, $\eta_t = 37.54038$ hours).

3. From Equation (30), the corresponding testing times are shown in Table 2.

In the Weibull analysis offered in Table 2, it is important to notice that columns λ_{1i} and λ_{2i} and S_i are all based on the vibration stress family W ($\beta s = 2.5$, $\eta s = 469.2544$ rms^2) and that the t_i column in Table 2 is based on the Weibull testing time family W ($\beta s = 2.5$, $\eta_t = 37.54038$ hours) also that from Equation (11), the values of the $R(t_i)$ and n_i columns are both completely related each other, and that the ($R(t_i)$ and n_i) columns are independent from the (λ_{1i}, λ_{2i}, S_i, and t_i) columns.

Thus, from Table 2, the original vibration testing design in Section 4.1 is given by combining the $R(t_i) = 0.97$ and $n_i = 33$ parts values of row between rows 1 and 2 with the $S = 2.76$ Grms and $t = 24$ hours values of row between rows 9 and 10. As a consequence, columns ($R(t)$ and n) are independent of the (S and t) columns, and the testing profile values appear in different rows. The next section describes the analysis by which all of them are set in the same row.

However, to show how the Weibull analysis shown in Table 2 can be used for designers and reliability practitioners, the following complement analysis will first be presented.

Practical complement analysis

First, it should be pointed out that from Equations (25) and (26), the values of column λ_{2i} in Table 2 can also be used to determine the reliability value, which corresponds to the $R(t_i)$ column given in Table 2 as follows:

$$R(t_i) = \exp\left\{-\left(\frac{\lambda_{2i}}{\eta_s}\right)^{\beta s}\right\} \tag{31}$$

Thus, from row between rows 9 and 10 of Table 2, it can be concluded that because from the applied testing profile shown in Table 1, the minimal applied stress is 300 Grms, then from Equation (31) the expected reliability as shown in Table 2 is of $R(t) = \exp\{-1 * (300/469.2544)^{2.5}) = 0.7212$. Similarly, from row between rows 1 and 2 of Table 2, it can be concluded that if the minimal applied stress is of 116.12 Grms, then as shown in Table 2, $R(t) = \exp\{-1 * (116.12/469.2544)^{2.5}) = 0.97$. Hence, since the real minimal applied stress is 300 Grms and the desired $R(t)$ index is $R(t) = 0.97$, then the strength of the product should be increased to increase the reliability from $R(t) = 0.7212$ to $R(t) = 0.97$.

By using data of Table 2, the minimal strength material to be used is selected based on the fact demonstrated in Piña-Monarrez,[7] where for higher $R(t)$ percentiles, (saying $R(t) > 0.90$), the column λ_{1i} of Table 2 can be used as the strength scale η_{Si} parameter of the Weibull strength distribution, which can be used to model the strength behavior of the used material. Therefore, by using the known $\beta s = 2.5$ value and the desired $\lambda_{1i} = \eta_{Si}$ value in Table 2 as the Weibull strength parameters W (βs, $\lambda_{1i} = \eta_{Si}$), the $R(t)$ index shown in Table 2 can also be determined by using the composed Weibull/Weibull (stress/strength) reliability function[21] given as follows:

$$R(t/s, S) = \int_0^\infty \left[\int_s^\infty f(S)dS\right] f(s)ds \tag{32}$$

which because in our case, βs is common for the stress W (βs, η_s) and the strength W (βs, $\lambda_{1i} = \eta_{Si}$) distributions, then from Piña-Monarrez,[7] the $R(t)$ index is given as follows:

$$R(t/s, S) = \frac{\eta_S^{\beta s}}{\eta_S^{\beta s} + \eta_s^{\beta s}} \tag{33}$$

Thus, since $R(t) = 0.97$ is desired, then from Table 2, $\lambda_{1i} = \eta_{Si} = 1896.32$ rms^2 is selected as the scale parameter of the Weibull strength distribution. Hence, with this value in Equation (33), $R(t) = 1896.32^{2.5}/(1896.32^{2.5} + 469.25^{2.5}) = 0.97$.

However, because from engineering handbooks, there was a search for average strength values, say yield strength Sy, and not for Weibull scale parameters, then by using the selected strength η_S value, the stress η_s value, and the minimum applied stress value, the minimum strength values, which should be searched for in the engineering handbook is given as follows:

$$\lambda_{S\min} = \frac{\lambda_{\min}\eta_S}{\eta_s}, \quad \lambda_{S\max} = \frac{\eta_s \eta_S}{\lambda_{\min}} \tag{34}$$

From Equation (34), by using $\eta_S = 1896.3223$ rms^2, $\eta_s = 469.2547$ rms^2, and $\lambda_{\min} = 300$ rms^2, the minimum product strength, which should be searched for, is $\lambda_{min} = 1212.34$ rms^2 (3.55 Grms). Thus, if the minimum product strength is of $\lambda_{\min} = 1212.34$ rms^2 (3.55 Grms), then its reliability will be of $R(t) = 0.97$.

Additionally, it should be noticed that because from Equation (34), the maximum expected strength is $\lambda_{\max} = 2966.19$, then since by using these minimum and maximum strength values in Equation (22), the estimated scale strength value is also $\eta_S = 1896.3223$ rms^2, then the whole theory explained in this paper could also be performed to the addressed strength Weibull family.

Following is the formulation to set the $R(t) = 0.97$, $n = 33$ parts, $S = 2.76$ Grms, and $t = 24$ hours values of the original testing profile in the same row of the Weibull analysis.

4.2.4 | Weibull testing plan analysis

This section is based on the normal vibration data given in the testing profile in Table 1 and Figure 1 in step 2 of Section 4.1. In addition, from the analysis above, in this section, it has been assumed that the tested product will withstand the applied vibration stress profile as mentioned in Practical complement analysis section. In other words, the assumption is that the tested product presents a minimum strength of $\lambda_{\min} = 1212.34$ rms^2, or equivalently, and that has been assumed because the strength of the tested product could be well represented by the strength Weibull distribution with parameters W ($\beta s = 2.5$, $\lambda_{1i} = \eta_{Si} = 1896.3223$ rms^2); thus, by applying the testing profile given in Table 1, no failures are expected.

As a result, in the Weibull analysis, the key vibration testing factors, $R(t) = 0.97$ sample size $n = 33$ parts, $t = 24$ hours, and $S = 2.76$ rms should be set at the row, which corresponds to $R(t) = 0.97$.

The steps to do this are given below.

Step 1. Following steps 1 to 6 of Section 4.2.2, determine the Y_i elements, the logarithm of the basic Weibull values, the basic Weibull values, the $R(t_i)$ indices and the n_i values.

Step 2. Determine the basic Weibull value which corresponds to the desired $R(t)$ index as follows:

$$\tan\left(\theta_{R(t)}\right) = \exp\left\{\frac{\ln(-\ln(R(t)))}{\beta_s}\right\} \tag{35}$$

Step 3. By using the S vibration level of step 3 of Section 4.2.1 (see Equation (18)) and the basic Weibull value of step 2, determine the vibration scale parameter in Grms units as follows:

$$\eta_G = \frac{S}{\tan\left(\theta_{R(t)}\right)} \tag{36}$$

Step 4. By using the η_G value from step 3 and the basic Weibull values from step 1, determine the vibration Grms levels as follows:

$$S_{Gi} = \eta_G{}^* \tan(\theta_i) \tag{37}$$

Step 5. By using the testing time t from step 1 in Section 4.2.1 and the basic Weibull value of step 2, determine the testing time scale parameter in hours as follows:

$$\eta_t = \frac{t}{\tan\left(\theta_{R(t)}\right)} \tag{38}$$

Step 6. By using the η_t value from step 5 and the basic Weibull values from step 1, determine the testing times as follows:

$$t_i = \eta_t{}^* \tan(\theta_i). \tag{39}$$

The next section describes the application.

Weibull testing's plan application

Here, vibration data from Section 4.2.1 are used. In the analysis from Equation (35), the basic Weibull value, which corresponds to $R(t) = 0.97$ is$\tan(\theta_{R(t)}) = 0.2474499$. Therefore, from Equation (36) $\eta_G = 11.153769$ Grms, and from

Equation (38) η_t = 96.989295hours. Table 3 shows the normal vibration analysis following the steps in Section 4.2.3. From Table 3, it can be observed that now all the original testing's parameters of the test IV of the ISO 16750-3 are in the $R(t) = 0.97$ row. This means that by testing $n = 33$ parts at constant vibration stress of $S = 2.76$ Grms by $t = 24$ hours each, the demonstrated reliability is $R(t) = 0.97$.

On the other hand, since now the values of columns S_{Gi} and t_i, represent the random behavior of the minimum applied stress and testing time of a product, which as expected present a $R(t) = 0.97$, then the values of these columns could be used to derive several accelerated testing plan scenarios such as those given in the next section.

4.2.5 | Weibull accelerated testing's plan analysis

This section is based on Equation (34), on the data from Table 3, and on the fact that from the data in Table 3, the ratio of the testing times and vibration levels between the fixed row of $R(t) = 0.97$ and any other row is the same, and it is given as follows:

$$\frac{t_n}{t_{ac}} = \frac{S_n}{S_{ac}} \tag{40}$$

where from Equation (18), S_n is the normal vibration level of the testing profile of $S = 2.76$ Grms, and t_n is the testing time of $t = 24$ hours; t_{ac} is the accelerated testing time, and S_{ac} is the accelerated vibration stress level. Therefore, from Equation (40), S_{ac} is given as follows:

$$S_{ac} = S_n\left(\frac{t_{ac}}{t_n}\right) = S_n{*}sf \tag{41}$$

Hence, from Table 3 or from Equations (12) and (40), it can be seen that the over stress factor is given as follows:

$$sf = \left(\frac{t_{ac}}{t_n}\right) = \left(\frac{S_{ac}}{S_n}\right) = \sqrt[\beta s]{\frac{n}{n_2}} \tag{42}$$

Therefore, based on the analysis from above, several testing scenarios are possible. The four most common are the following:

S1. Testing fewer parts for the given vibration level S, but at an accelerated testing time t_{ac}. In this case, S is a fixed value, and t_{ac} ($t_{ac} > t$). For the selected n_2 ($n_2 < n$) value, it is given from Equations (41) and (42). For example, with fixed $S = 2.76$ Grms value, the t_{ac} value that corresponds to $n_2 = 10$ is, $t_{ac} = 24 * (32.83/10)^{0.4} = 24 * 1.60883 = 38.61$ hours.

Thus, by testing $n_2 = 10$ parts for $t_{ac} = 38.61$ hours, each at constant vibration level of $S = 2.76$ Grms, $R(t) = 0.97$ can be demonstrated. (The same fact is true for any desired row of Table 3). Here, it can be noticed that since S is a fixed value, then the testing profile is the one shown in Table 1 and in Figure 1.

S2. Testing more parts for the given vibration level S, but for a shorter testing time. In this case S is a fixed value, and the testing's time t_i ($t_i < t$). For the selected n_2 ($n_2 > n$) value, it is given from Equations (41) and (42). For example, with fixed $S = 2.76$ Grms value, the t_i value that corresponds to $n_2 = 47.21$ parts is, $t_i = 24 * (32.83/47.021)^{0.4} = 24 * 0.8662 = 20.75$ hours.

Thus, by testing $n_2 = 47.21$ parts for $t_i = 20.75$ hours each at constant vibration level of $S = 2.76$ Grms, $R(t) = 0.97$ can be demonstrated. (The same fact is true for any desired row of Table 3). Here, it can be noticed that since S is the fixed value, then the testing profile is the one given in Table 1 and in Figure 1.

S3. Testing fewer parts for the given testing time t, but at an accelerated vibration level S_{ac}. In this case, t is a fixed value, and the S_{ac} ($S_{ac} > S$) level. For the selected n_2 ($n_2 < n$) value, it is given from Equations (41) and (42). For example, from Equation (41), with fixed $t = 24$ hours, the S_{ac} level that corresponds to $n_2 = 10$, is $S_{ac} = 2.76 * (32.83/10)^{0.4} = 2.76 * 1.60883 = 4.44$ Grms.

Thus, by testing $n_2 = 10$ parts for $t = 24$ hours each at constant accelerated vibration level of $S_{ac} = 4.44$ Grms, $R(t) = 0.97$ can be demonstrated. (The same fact is true for any desired row of Table 3). The corresponding accelerated testing's profile is given in Table 4 and in Figure 2.

In Table 4, the accelerated frequencies were determined by using the sf factor of Equation (42) as follows:

$$f_{ac} = f_i * sf^2 \tag{43}$$

Note . It is worth pointing out that under equipment capability restriction, instead of accelerating the frequency as in Equation (43), the PSD energy could be equivalently accelerated by replacing f_i by the corresponding PSD_i value as $PSD_{ac} = PSD_i * sf^2$.

S4. Testing more parts for the given testing time t, but at a lower vibration level S. In this case, t is a fixed value, and the vibration level S_i ($S_i < S$). For the selected n_2 ($n_2 > n$) value, it is given from Equations (41) and (42). For example, with fixed $t = 24$ hours, the S_i value that corresponds to $n_2 = 47.21$ is, $S_i = 2.76 * (32.83/47.021)^{0.4} = 2.76 * 0.8661 = 2.39$ Grms.

Thus, by testing $n_2 = 47.21$ parts for $t = 24$ hours each at constant vibration level of $S = 2.39$ Grms, $R(t) = 0.97$ can be demonstrated. (The same fact is true for any desired row of Table 3). The testing profile is given in Table 5 and in Figure 3.

As a summary of this section, notice that the above analysis was performed based on the row in Table 3, which contains the testing factors of the used profile. The same analysis will be presented next but by considering the fatigue exponent M of the used material.

TABLE 4 ISO16750-3, test IV: Accelerated random vibration profile

Freq, Hz	PSD $(m/s^2)^2$/Hz	dB	Oct	dB/Oct	Area	rms
25.88	30.0000	*	*	*	776.51	27.87
1035.35	0.2000	−21.76	5.32	−4.09	1589.25	39.87
2588.36	0.2000	0.00	1.32	0.00	1899.85	**43.59**

* it only indicates it is an empty cell.

Bold number shown readers the total accumulated energy.

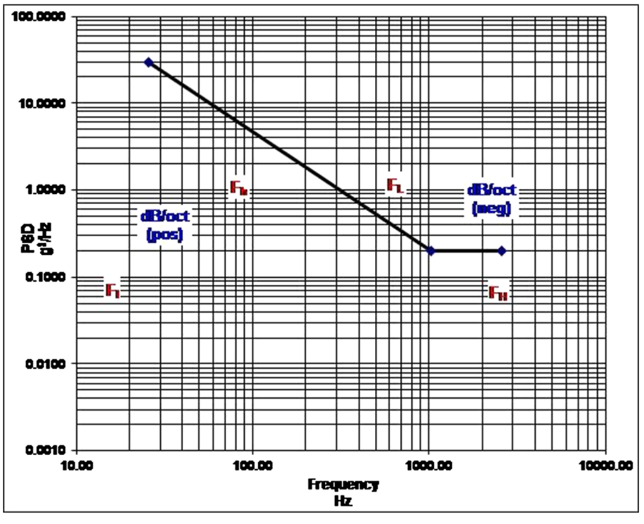

FIGURE 2 ISO16750-3, accelerated testing's profile [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 5 ISO16750-3, test IV: Accelerated random vibration profile

Freq, Hz	PSD $(m/s^2)^2$/Hz	dB	Oct	dB/Oct	Area	rms
7.50	30.0000	*	*	*	225.07	15.00
300.09	0.2000	−21.76	5.32	−4.09	460.64	21.46
750.23	0.2000	0.00	1.32	0.00	550.66	**23.47**

* it only indicates it is an empty cell.

Bold number shown readers the total accumulated energy.

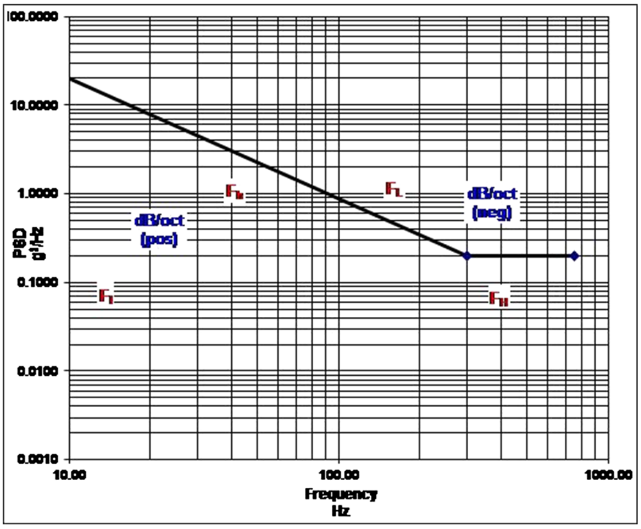

FIGURE 3 ISO16750-3, accelerated testing's profile [Colour figure can be viewed at wileyonlinelibrary.com]

4.2.6 | Weibull accelerated fatigue testing's plan analysis

From appendix I of the user guide for the norm GMW3172, page 299, the fatigue exponent M of the used material is incorporated in the analysis. It is incorporated in the stress factor defined in Equation (42). Therefore, the corresponding fatigue stress factor is given as follows:

$$sfm = \left(\frac{S_{ac}}{S_n}\right) = \sqrt[M]{\frac{t_{ac}}{t_n}} = \sqrt[\beta sM]{\frac{n}{n_2}} \tag{44}$$

As a result, from Equations (41) and (44), the accelerated fatigue vibration level is given as follows:

$$S_{af} = S_n\left(\frac{S_{ac}}{S_n}\right)^{1/M} \tag{45}$$

Now, by considering that the tested product is made of aluminum lead, and according to appendix I of the user guide for norm GMW3172, page 299, the M fatigue coefficient is $M = 6.4$. Thus, the fatigue vibration testing analysis for the four scenarios in section 4.2.6.3 is as follows.

SF1. Testing fewer parts for the given vibration level S, but at an accelerated testing's time. In this case, from Equation (45) $S_{ac} = S$, then $S_{af} = Sn = 2.76$ Grms. Consequently, this scenario is not affected by the M coefficient. Here, it can be noticed that since S_{af} is a fixed value, then the testing's profile is the one given in both Table 1 and in Figure 1.

SF2. Testing more parts for the given vibration level S, but at a lower testing time. This scenario is not affected by the M coefficient either. Here, it can be noticed that since S_{af} is a fixed value, then the testing profile is the one given in both Table 1 and in Figure 1.

SF3. Testing fewer parts for the given testing time t, but at an accelerated vibration level. In this case, by using $S_{ac} = 4.44$ Grms, $S_n = 2.76$ Grms, and $M = 6.4$ in Equation (45), $S_{af} = 2.97$ Grms. Thus, by testing $n_2 = 10$ parts

TABLE 6 ISO16750-3, test IV: Accelerated random vibration profile

Freq, Hz	PSD $(m/s^2)^2$/Hz	dB	Oct	dB/Oct	Area	rms
11.60	30.0000	*	*	*	348.06	18.66
464.08	0.2000	−21.76	5.32	−4.09	712.36	26.69
1160.21	0.2000	0.00	1.32	0.00	851.59	**29.18**

* it only indicates it is an empty cell.

Bold number shown readers the total accumulated energy.

for t = 24 hours each at constant accelerated vibration level of, S_{af} = 2.97 Grms, $R(t)$ = 0.97 is demonstrated. (The same fact is true for any desired row of Table 3). The corresponding accelerated testing profile is given in Table 6 and in Figure 4. In Table 6, the accelerated fatigue frequencies were determined by using the *sf* factor of Equation (42) as follows:

$$f_{ac} = f_i * sf^{2/M} \tag{46}$$

Note . As can be observed instead of accelerating the frequency, the PSD energy could be accelerated as $PSD_{ac} = PSD_i * sf^{2/M}$.

SF4. Testing more parts for the given testing time t, but at a lower vibration level S. In this case, by using Sac = 2.39 Grms, Sn = 2.76 Grms, and M = 6.4 in Equation (45), Saf = 2.70 Grms. Thus, by testing n_2 = 47.21 parts for t = 24 hours each at constant accelerated vibration level of, Saf = 2.70 Grms, $R(t)$ = 0.97 can be demonstrated. (The same fact is true for any desired row in Table 3). The corresponding accelerated fatigue testing's profile is given in Table 7 and in Figure 5.

Finally, the Weibull/Fatigue family which could be used in the corresponding fatigue analysis will be derived.

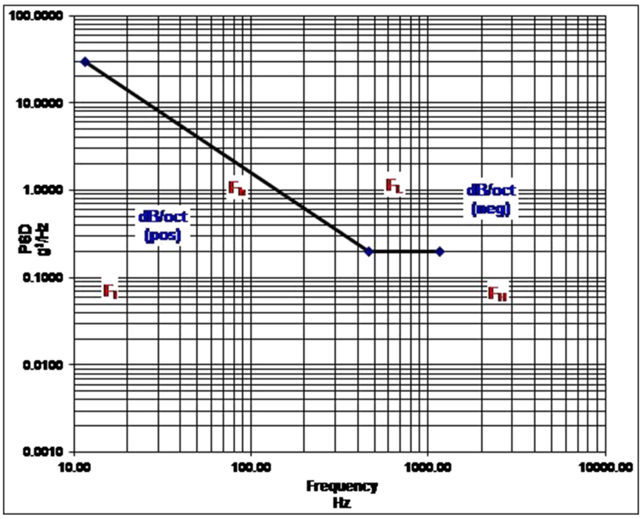

FIGURE 4 Accelerated fatigue testing profile [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 7 ISO16750-3, test IV: Accelerated random vibration profile

Freq, Hz	PSD $(m/s^2)^2$/Hz	dB	Oct	dB/Oct	Area	rms
9.56	30.0000	*	*	*	286.83	16.94
382.44	0.2000	−21.76	5.32	−4.09	587.04	24.23
956.09	0.2000	0.00	1.32	0.00	701.77	**26.49**

* it only indicates it is an empty cell.

Bold number shown readers the total accumulated energy.

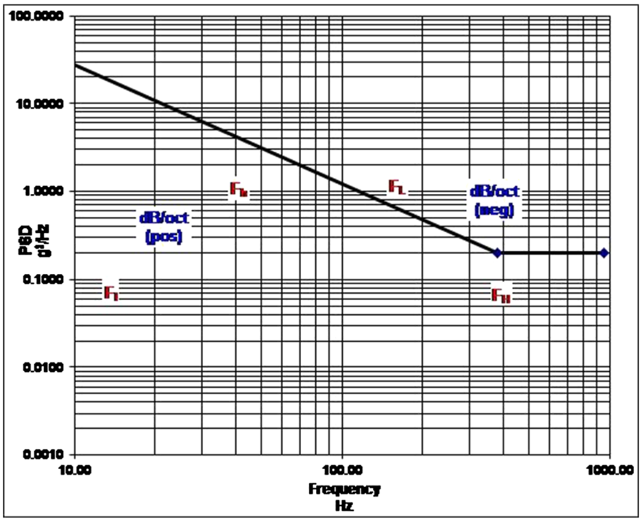

FIGURE 5 Accelerated fatigue testing profile [Colour figure can be viewed at wileyonlinelibrary.com]

4.3 | Weibull/fatigue relationships

By using the material fatigue exponent M, the Weibull shape fatigue parameters β_f is given as follows:

$$\beta_f = \beta_s{*}M = \frac{\sigma_y}{\sigma_S} \tag{47}$$

And by using the corresponding vibration level S of the used profile and the above β_f value in Equation (38), the Weibull scale fatigue parameter η_f

$$\eta_f = S/\exp\left\{\frac{\ln(-\ln(R(t))}{\beta_f}\right\} \tag{48}$$

Numerically, $\beta_f = 2.5 * 6.4 = 16$, and $\eta_f = 2.76 * \exp\{ln(-ln(0.97))/16\} = 3.433023$ Grms. Therefore, the corresponding Weibull fatigue stress family is W ($\beta_f = 16$, $\eta_f = 3.433023$ Grms).

Additionally, based on Equations (47) and (48), the Weibull scale parameter, which is used to determine the mean and the amplitude fatigue stresses, is given as follows:

$$\eta_{ft} = \exp\left\{\eta_f - \frac{\mu_y}{\beta_f}\right\} \tag{49}$$

Numerically, it is $\eta_{ft} = \exp\{3.433023\text{-}(-0.55479/16)\} = 3.554149$ Grms. Thus, by using the η_{ft} value and the basic fatigue Weibull values given by:

$$\tan(\theta_{fi}) = \frac{Y_i}{\beta_f} \tag{50}$$

the S_{fti} values are generated as follows:

$$S_{fti} = \eta_{ft}{*}\tan(\theta_{fi}) \tag{51}$$

and by taking the arithmetic mean of theS_{fti} values as the mean stress μ, the principal stresses values are given as follows:

$$\sigma_1, \sigma_2 = \mu \pm \sqrt{\mu^2 - \eta_f^2} \tag{52}$$

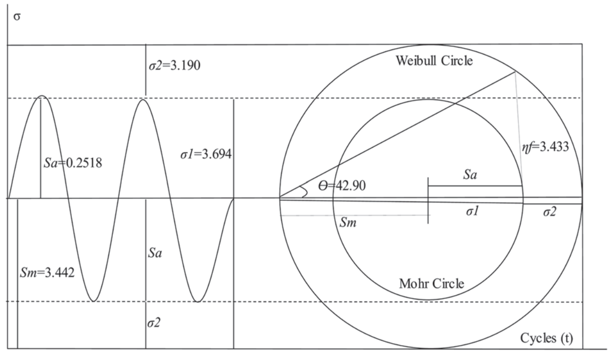

FIGURE 6 Weibull/fatigue representation

Numerically, $\mu = 3.442251$ Grms. Hence, from Equation (52), $\sigma_1 = 3.442251 + 0.251883 = 3.694134$ Grms, and $\sigma_2 = 3.190368$ Grms. As a consequence, the amplitude fatigue stress value is given as follows:

$$S_a = \frac{\sigma_1 - \sigma_2}{2}. \tag{53}$$

Numerically it is $S_a = 0.251883$ Grms, and the corresponding basic Weibull angle θ is as follows:

$$\theta_i = tg^{-1}\left(\sqrt{\frac{\sigma_{2i}}{\sigma_{1i}}}\right) = tg^{-1}\left(\frac{1}{\sqrt[\beta]{n_{2i}}}\right) = tg^{-1}(S_{0i}). \tag{54}$$

Numerically, it is $\theta = \text{tg}^{-1}((3.190368/3.694134)^{0.05}) = 42.90$. Therefore, from the above data, the cyclical Weibull/fatigue stress behavior in Grms units is shown in Figure 6.

As can be seen in Figure 6, if we know the ultimate strength S_{uG}, the yield strength S_{yG}, and the endurance limit S_{eG}, of the material in Grms units, the modified Goodman diagram analysis could be performed also, but because they are unknown, then more research should be undertaken. However, as Liou et al[22] mentions, because from the fatigue stress distribution W (β_f, η_f), the relationship between S_i and t_i (see Equations 48 and 49) as well as the cumulated damage (see Equation (2)) are all given; thus, from the previous analysis, it seems to be possible to derive formulas to estimate not only the cumulated damage, as it is done in section 2 in Liou et al[22] but also to predict the fatigue life based on the given σ_1 and σ_2 values, as it is given in Castillo et al,[23] or by using the first-order reliability method given in Xiang and Liu,[24] but because we do not know S_{uG}, S_{yG}, and S_{eG}, then more research should be undertaken. Following are the general conclusions of this work.

5 | CONCLUSIONS

1. From Equation (13), it is concluded that by applying the standard vibration test with $n = 23$ parts for $t = 24$ hours each, the demonstrated reliability is always lower than $R(t) = 0.97$ ($0.9574 < 0.97$). Because in this standard vibration test no failure time is available, then neither the probabilistic behavior of the applied vibration profile nor the fatigue analysis can be performed.
2. From Equation (11), the right n value, which completely represents both the desired $R(t)$ index and the related Weibull scale parameter value can be estimated.
3. Since the n value defined in Equation (11) always represents $R(t)$ and η, then by using it in Equation (30) the basic Weibull elements to determine the probabilistic behavior of both S and t can always be performed.
4. Since in the analysis the Weibull parameters are estimated directly from the testing profile, then the analysis completely represents the used testing profile.
5. The minimum strength (rms^2) that the product should present to withstand the applied stress with the desired reliability will be determined from the Weibull analysis shown in Table 2.
6. From the Weibull analysis given in Table 3, several accelerated testing plans scenarios to test fewer parts for a longer time and vice versa can be determined, and the same can be done for the fatigue analysis.

7. By applying the proposed Weibull analysis, it is possible to determine the midrange, the alternating, and the principal stresses values to perform the corresponding fatigue analysis. However, since there were no vibration data in rms^2 in the engineering handbooks used, more research is necessary in this area.
8. If the fatigue exponent is known, it can be incorporated in the Weibull analysis in the derived over-stress factor. It should be noticed here that the Weibull stress shape parameter and the Weibull fatigue shape parameters are related[25] as in Equation (47).
9. On the other hand, it is important to mention that if the time Weibull family W (β_t, η_t) or the stress Weibull family W (βs, ηs) are determined based on several stress variables, then the Taguchi method given in[26] can be used to determine the corresponding Weibull parameters.
10. Finally, it should be considered for further research that because n in Equation (11) is the reciprocal of the cumulative risk function $H(t)$, and since $H(t)$ is the mean of the related non-homogeneous Poisson processes, referred to in literature as the Weibull process (see Rinne[27] section 4.2.6, page 199), then it seems possible, by setting the $H(t)$ value as the critical cumulated damage, to use Equation (11) to estimate the n expected shocks of the related additive cumulative damage model defined in Nakagawa[28]; however, more research should be undertaken in this area.

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