Received: 25 April 2016 | Revised: 5 March 2018 | Accepted: 6 March 2018
DOI: 10.1002/asmb.2329

RESEARCH ARTICLE

WILEY

Degradation modeling of 2 fatigue-crack growth characteristics based on inverse Gaussian processes: A case study

Luis Alberto Rodríguez-Picón[1] | Anna Patricia Rodríguez-Picón[2] | Alejandro Alvarado-Iniesta[1]

[1]Department of Industrial Engineering and Manufacturing, Autonomous University of Ciudad Juarez, Ciudad Juarez, Mexico

[2]Post Graduate and Research Studies, Technological Institute of Ciudad Juarez, Ciudad Juarez, Mexico

Correspondence
Luis Alberto Rodríguez-Picón, Department of Industrial Engineering and Manufacturing, Autonomous University of Ciudad Juarez, Av. del Charro 450 norte, Zona Pronaf Juarez Chihuahua Mexico 32310.
Email: luis.picon@uacj.mx

Abstract

Most modern products that are highly reliable are complex in their inner and outer structures. This situation indicates quality characterization by the interaction of multiple performance characteristics, which motivates the utilization of robust reliability models to obtain robust estimates. It is paramount to obtaining substantial information about a product's life cycle; therefore, when multiple performance characteristics are dependent, it is important to find models that address the joint distribution of performance degradation of such. In this paper, a reliability model for products with 2 fatigue-crack growth characteristics related to 2 degradation processes is developed. The proposed model considers the dependence among degradation processes by using copula functions considering the marginal degradation processes as inverse Gaussian processes. The statistical inference is performed by using a Bayesian approach to estimate the parameters of the joint bivariate model. A time-scale transformation is considered to assure monotone paths of the degradation trajectories. The comparison results of the reliability analysis, under both dependent and independent assumptions, are reported with the implementation of the proposed modeling in a case study, which consists of the crack propagation data of 2 terminals of an electronic device.

KEYWORDS

Bayesian inference, copula function, degradation, fatigue-crack, inverse Gaussian process

1 | INTRODUCTION

Degradation information of specific characteristics are obtainable for most products. This information can be useful to make inferences about the life of the product.[1] Although the obtained information may be different depending of the product under study, in some, it is possible to measure the physical degradation as a function of time. In others, instead, measures of the performance degradation of the product may be available. Meeker and Escobar[2] presented examples in which degradation processes are modeled, taking into consideration the effect of accelerating variables, and described specific models for degradation curves. Usually, these models start with a deterministic description of the degradation process, and after this, randomness is introduced to describe the variability in initial conditions and parameters of the model as the product properties.

There are several classes of degradation models that have been comprehensively studied and applied to the reliability analysis of degradation data. The gamma process has been identified as a good model for degradation processes given the characteristics that its increments are independent and nonnegative, having a gamma distribution with an identical scale parameter.[3-6] Moreover, considering its monotonicity property and that performance can only decrease over time, this is why it is considered to be suitable to model wear, crack growth, corrosion, consumption, fatigue, etc.[7-13] Although the use of gamma process may be complicated when dealing with the first-time passage distributions, given that the obtained probability distribution function (PDF) has no explicit form. This implies the use of approximations to the Birnbaum-Saunders distribution and the inverse Gaussian distribution, which consider a discrete version of the first passage times of the gamma process and the central limit theorem to approach the passage time of the normalized cumulative degradation increments with a critical level to the Birnbaum-Saunders distribution.[8,14,15]

The geometric Brownian motion as a degradation process has been presented as another important alternative. The sample path of this process, as in the case of the gamma process, is also monotone. Different applications of this process in degradation analysis can be found in the works of Park and Padgett,[9] Elsayed and Liao,[16] Park and Padgett,[17] and Chiang et al.[18]

The inverse Gaussian (IG) process is another important choice for degradation modeling; this process also provides monotone degradation paths, and it is closely related to the Wiener process with drift. Indeed, it can be treated as the first passage distribution of a Wiener process. An important practical advantage of the IG process over the gamma process is the closed form of its first-time passage distribution. In addition, as described by Ye and Chen,[19] this process is flexible when the incorporation of random effects and covariates is of interest in the modeling of degradation process, which accounts for heterogeneities. Several recent studies have been presented considering the IG process as a degradation model; Wang and Xu[20] presented a method for incorporating random effects and covariates in the IG process and presented a scheme for maximum likelihood estimation (MLE) on the basis of the Expectation-Maximization (EM) algorithm. Ye and Chen[19] extended the method proposed by Wang and Xu[20] by proposing 2 different approaches to incorporate random effects in the IG process. The first approach consisted in letting the drift parameter of the IG process be a random variable and assuming that the inverse drift parameter follows a truncated normal distribution, and the second approach consisted in letting the volatility parameter of IG be a random variable and that follows a gamma distribution. Peng[21] introduced a new model based on the inverse normal-gamma mixture of an IG process; in this model, the parameters of the models can vary from unit to unit by using the natural conjugate distribution for IG data.

Given that most of the modern highly reliable products are complex in their inner and outer structures, the degradation research topic continues to draw great interest. Considering that, the quality of a certain product might be characterized by the interaction of multiple performance characteristics (PC), which may be dependent. Then, it is important to consider the joint distribution of the multiple degradation processes. The use of copula functions is an appealing way for dealing with multivariate distributions constructed from marginal densities. Different ways of using copula functions with degradation data can be found in the works of Liu et al,[22] Sari et al,[23] Pan et al,[24] and Wang and Pham.[25] Liu et al[22] proposed a bivariate degradation model on the basis of IG processes with s-dependence between the marginal degradation processes and the random drift model proposed by Ye and Chen.[19] They also presented the joint modeling of the IG processes via copula functions and MLE estimations of the parameters via the 2-stage EM algorithm. Although MLE is an important estimation method, it may have some disadvantages when dealing with degradation data, as mentioned by Peng et al,[26] for example, few availability of degradation measures and the no possibility of incorporating historical information about the data into the modeling. Peng et al[26] presented different Bayesian schemes for the estimation of a univariate IG process with random effects.

In this paper, the degradation modeling of 2 PC considering 2 IG processes via copula functions is considered. As the bivariate joint distributions are complex, the estimation of the parameters is performed via the Gibbs sampling and Markov chain Monte Carlo (MCMC) simulation. Firstly, a simulation study considering the proposed estimation approach and the MLE are carried out. A bivariate model based on IG processes and a copula function is considered to illustrate the advantages of the Bayesian approach over the MLE. The 2 schemes are compared on the basis of the mean square error of the estimated parameters. Secondly, the proposed model is implemented in a case study that consists of crack propagation data of 2 terminals of an electronic device. As part of the estimation process, we considered a second run of the Gibbs sampling by taking the posterior information obtained of the first run as prior information for the second run. In addition, the fitting of the IG process in the case study is evaluated by also fitting the case study degradation data set to the gamma and geometric Brownian motion processes. Considering the best fitting estimations, the reliability assessment of the device under study is performed by assessing the dependence of the terminals.

The rest of the paper is organized as follows. In Section 2, the univariate IG process with time-scale transformation is presented. In Section 3. The bivariate modeling of the 2 IG processes is introduced considering copula functions. In Section 4, the inference method based on Gibbs sampling is presented. In addition, a simulation study is performed to compare the performance of the Gibbs sampling and the MLE. In Section 5, the case study is presented to illustrate the proposed models. The IG model is compared with other stochastic processes with the aim to determine the best-fitting model. In addition, the reliability assessment is provided. In Section 6, the concluding remarks and discussion are provided.

2 | IG PROCESS WITH TIME-SCALE TRANSFORMATION

Before the bivariate IG process is defined, we first introduce the marginal IG processes. Each one of these processes represents the degradation of a PC of the device under test. We consider that the degradation path over time is modeled by a stochastic process $\{Z(t); t > 0\}$. Specifically, it is considered that an IG process governs the degradation process of a PC. The IG process with drift parameter (μ) and diffusion parameter (λ) has the following characteristics,

- $Z(t) - Z(s) = \Delta Z(t)$ follows an IG distribution IG $(\mu[\tau(t) - \tau(s)], \lambda[\tau(t) - \tau(s)]^2)$.
- $Z(t)$ has independent increments, ie, $Z(t_4) - Z(t_3)$ and $Z(t_2) - Z(t_1)$ are independent, $\forall t_1 < t_2 < t_3 < t_4$,

where, $\tau(t)$ is a monotone increasing function. In this case, $\tau(t)$ is considered as a monotone time-scale transformation $\tau(t) = \tau(t, \gamma)$ with the form $\tau = t^{\gamma}$, given that this transformation can be used when dealing with power relationships[27] such as the case study presented in this paper. Considering that, $\Delta Z(t)$ is governed by IG $(\mu\tau(t,\gamma), \lambda[\tau(t,\gamma)]^2)$ with mean $\mu\tau(t,\gamma)$ and variance $\mu^3\tau(t,\gamma)/\lambda$ and has the following PDF,

$$f_{\mathcal{IG}}(\Delta Z(t)) = \sqrt{\frac{\lambda\tau^2(t,\gamma)}{2\pi\Delta Z^3(t)}} \exp\left\{-\frac{\lambda(\Delta Z(t) - \mu\tau(t,\gamma))^2}{2\mu^2\Delta Z(t)}\right\} \tag{1}$$

and cumulative distribution function (CDF)

$$F_{\mathcal{IG}}(\Delta Z(t)) = \Phi\left[\sqrt{\frac{\lambda}{\Delta Z(t)}}\left(\frac{\Delta Z(t)}{\mu} - \tau(t,\gamma)\right)\right] + \exp\left\{\frac{2\lambda\tau(t,\gamma)}{\mu}\right\} \times \Phi\left[-\sqrt{\frac{\lambda}{\Delta Z(t)}}\left(\frac{\Delta Z(t)}{\mu} + \tau(t,\gamma)\right)\right], \tag{2}$$

where Φ is the standard normal cumulative distribution function.

As the degradation path of a product's PC is governed by IG $(\mu\tau(tA\gamma), \lambda[\tau(t,\gamma)]^2)$, then when the path reaches a critical degradation value ω, the lifetime T_ω of the product is defined as $T_\omega = \inf\{Z(t) \geq \omega\}$. Given the monotonicity property of the IG process, it is well known[19,20] that the reliability function of the product under study can be obtained as

$$R(t) = \Phi\left[\sqrt{\frac{\lambda}{\omega}}\left(\frac{\omega}{\mu} - \tau(t,\gamma)\right)\right] + \exp\left\{\frac{2\lambda\tau(t,\gamma)}{\mu}\right\} \times \Phi\left[-\sqrt{\frac{\lambda}{\omega}}\left(\frac{\omega}{\mu} + \tau(t,\gamma)\right)\right]. \tag{3}$$

Now, consider that a product has 2 PCs and that they are marginally governed by an IG process with the previously described time-scale transformation. During a degradation test, N units are tested, and M measurements for all the units are observed up to the termination time T, which results in degradation measurements $Z_{ik}(t_j)$ of the ith unit at the corresponding time t_j, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, and $k = 1, 2$ PC. Then, the degradation data can be presented as follows

$$X_{2N\times M} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_{11}(t_1) & \cdots & Z_{11}(t_M) \\ \vdots & \ddots & \vdots \\ Z_{N1}(t_1) & \cdots & Z_{N1}(t_M) \\ Z_{12}(t_1) & \cdots & Z_{12}(t_M) \\ \vdots & \ddots & \vdots \\ Z_{N2}(t_1) & \cdots & Z_{N2}(t_M) \end{pmatrix}. \tag{4}$$

According to the independent increment property of the IG process and $\Delta Z_{ik}(t_j) = Z_{ik}(t_j) - Z_{ik}(t_{j-1})$, $t_0 = 0$, $\Delta\tau\left(t_j, \gamma_k\right) = \tau\left(t_j, \gamma_k\right) - \tau\left(t_{j-1}, \gamma_k\right) = t_j^{\gamma_k} - t_{j-1}^{\gamma_k}$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, and $k = 1, 2$ PC. Thus, it is possible to obtain independent random variables

$$\Delta Z_{ik}(t_j) \sim IG\left(\mu_k\Delta\tau\left(t_j, \gamma_k\right) \lambda_k\Delta\tau^2\left(t_j, \gamma_k\right)\right).$$

The PDF of $\Delta Z_{ik}(t_j)$ is defined as follows:

$$f_{\mathcal{IG}}\left(\Delta Z_{ik}(t_j)\right) = \sqrt{\frac{\lambda_k\Delta\tau^2\left(t_j, \gamma_k\right)}{2\pi\Delta Z_{ik}^3(t_j)}} \exp\left\{-\frac{\lambda_k\left(\Delta Z_{ik}(t_j) - \mu_k\Delta\tau\left(t_j, \gamma_k\right)\right)^2}{2\mu_k^2\Delta Z_{ik}(t_j)}\right\}, \tag{5}$$

where, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, and $k = 1, 2$.

If ω_k is the critical degradation level of each PC for $k = 1, 2$, then the CDF of T_{ω_k} can be obtained from Equation (3) as $P(Z_{ik}(t_j) \geq \omega_k)$.

3 | BIVARIATE MODELING VIA COPULA FUNCTIONS

Copulas are parametric functions that join univariate distributions into multivariate distribution functions. Most of them have convenient parametric forms, which allows the modeling of the dependence structure among marginal distribution functions.[28] The copula approach to model dependence is linked to the Sklar theorem representation[29]

$$H(x, y) = C\left\{F(x), G(y)\right\}, \quad x, y \in \mathbb{R}$$
$$F \in F_{\varphi}, \quad G \in G_{\omega}, \quad C \in C_{\theta},$$

where, $F(x)$, $G(y)$ are marginal CDFs and x, y are random variables. If $H(x, y)$ is a joint distribution with marginal distributions $F(x)$ and $G(y)$, then a unique copula C can be obtained for x, y. Thus, a bivariate copula is a CDF defined in $[0, 1]^2$ with uniform marginal distributions $[0, 1]$. There are different copula functions with one parameter that can be used such as the Archimedean family.[30] In this case, some Archimedean copulas are considered to jointly model the 2 degradation processes $\Delta Z_{i1}(t_j)$ and $\Delta Z_{i2}(t_j)$ as follows:

$$H\left(\Delta Z_{i1}, (t_j)\Delta Z_{i2}(t_j)\right) = C\left(U_{ij1}, V_{ij2}\right),$$

where $U_{ij1} = F_{\mathcal{IG}}\left(\Delta Z_{i1}(t_j)\right)$ and $V_{ij2} = F_{\mathcal{IG}}\left(\Delta Z_{i2}(t_j)\right)$ are defined on the basis of Equation (2)

$$F_{\mathcal{IG}}\left(\Delta Z_{ik}(t_j)\right) = \Phi\left[\sqrt{\frac{\lambda_k}{\Delta Z_{ik}(t_j)}}\left(\frac{\Delta Z_{ik}(t_j)}{\mu_k} - \Delta\tau\left(t_j, \gamma_k\right)\right)\right] + \exp\left\{\frac{2\lambda_k \Delta\tau\left(t_j, \gamma_k\right)}{\mu_k}\right\} \times \Phi\left[-\sqrt{\frac{\lambda_k}{\Delta Z_{ik}(t_j)}}\left(\frac{\Delta Z_{ik}(t_j)}{\mu_k} + \Delta\tau\left(t_j, \gamma_k\right)\right)\right], k = 1, 2.$$

Considering any of the copula functions from the Archimedean family with one parameter described in Table 1 as $c(u, v; \theta) = \partial C(U_{ij1}, V_{ij2})/\partial U_{ij1}\partial V_{ij2}$ to be the bivariate PDF of the 2 degradation processes. Then, the likelihood and log-likelihood functions are described as

$$L(\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta) = \prod_{i=1}^{N}\prod_{j=1}^{M}\left[c(U_{ij1}, V_{ij2}; \theta) \cdot f_{\mathcal{IG}}(\Delta Z_{i1}(t_j)) \cdot f_{\mathcal{IG}}(\Delta Z_{i2}(t_j))\right] \tag{6}$$

$$l\left(\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta\right) = \sum_{i=1}^{N}\sum_{j=1}^{M}\left[\ln\left(c(U_{ij1}; V_{ij2}; \theta)\right) + \sum_{k=1}^{2}\ln\left(f_{\mathcal{IG}}\left(\Delta Z_{ik}(t_j)\right)\right)\right]. \tag{7}$$

The product is considered to have failed if any of the PC reaches the critical degradation level ω_k, $k = 1, 2$. Thus, the reliability function can be described as

$$R(t) = P\left\{Z(t_1) < \omega_1, Z(t_2) < \omega_2\right\} = C\left(P\left(Z(t_1) < \omega_1\right), P\left(Z(t_2) < \omega_2\right)\right). \tag{8}$$

TABLE 1 One parameter bivariate copulas

Copula	$C(u, v)$	Parameter
Plackett	$1 + (\theta - 1)(u + v) - \sqrt{1 + (\theta - 1)(u + v)^2 + 4\theta(1 - \theta)}/(\theta - 1)/2$	$\theta \geq 0$
Frank	$\theta^{-1} \ln\left[1 + (\exp\{\theta u\} - 1)(\exp\{\theta v\} - 1)(\exp\{\theta\} - 1)^{-1}\right]$	$\theta \neq 0$
Gumbel	$\exp\{-[(-(\ln u)^{\theta} + (-\ln v)^{\theta})]^{1/\theta}\}$	$\theta \geq 1$
Clayton	$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$	$\theta > 0$
Joe	$1 - [(1 - u)^{\theta} + (1 - v)^{\theta} - (1 - u)^{\theta}(1 - v)^{\theta}]^{1/\theta}$	$\theta \geq 1$
AMH	$uv/1 - \theta(1 - u)(1 - v)$	$-1 \leq \theta \leq 1$

4 | ESTIMATION OF PARAMETERS BASED ON A BAYESIAN APPROACH

Considering that $Z_{ik}(t_j)$ for $k = 1, 2$ have been observed for N units at time points t_j for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, then the degradation increments $\Delta Z_{ik}(t_j)$ are independent random variables that follow IG distributions as $IG\ (\mu_k \Delta\tau(t_j, \gamma_k), \lambda_k \Delta\tau^2(t_j, \gamma_k))$. The bivariate models are constructed by using the different copula functions described in Table 1 for the marginal IG processes $k = 1, 2$. The Bayesian approach is implemented by considering the following scheme of prior distributions: noninformative prior distributions are considered for the parameters of interest $\delta = (\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta)$. In general, truncated normal prior distributions $TN\left(a_{\mu_k}, b_{\mu_k}\right)$ with mean hyperparameter a_{μ_k} and scale hyperparameter b_{μ_k} are considered for μ_k, $k = 1, 2$, to avoid negative values of the mean parameters of the marginal IG processes. A gamma prior distribution $Ga\left(\vartheta_{\lambda_k}, \kappa_{\lambda_k}\right)$ with shape hyperparameter ϑ_{λ_k} and scale hyper-parameter κ_{λ_k} are considered for λ_k, $k = 1, 2$, given that the gamma distribution is the conjugate prior distribution of λ_k. Noninformative normal distributions $N\left(a_{\gamma_k}, b_{\gamma_k}\right)$ with mean hyper-parameter a_{γ_k} and scale hyper-parameter b_{γ_k} are considered for γ_k, $k = 1, 2$. For the association parameter (θ) of the copula function, the prior distribution is defined depending of the considered Archimedean copula. For example, for the Frank copula, a noninformative normal distribution $N(a_\theta, b_\theta)$ is considered. For the Gumbel and Joe copulas, a noninformative uniform distribution $U(c_\theta, d_\theta)$ with $c_\theta = 1$ is considered. For the Clayton and Plackett copulas, a noninformative uniform distribution $U(c_\theta, d_\theta)$ with $c_\theta = 0$ is considered. For the AMH copula, a noninformative uniform distribution $U(c_\theta, d_\theta)$ with $c_\theta = -1$ and $d_\theta = 1$ is considered.

Considering the previously described a priori distributions for $\delta = (\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta)$ and the likelihood function described in (6), the joint posterior distribution can be expressed as

$$p\left(\delta; \Delta Z_{ik}(t_j)\right) \propto f_{TN}\left(\mu_1\right) \cdot f_{TN}(\mu_2) \cdot f_{\mathcal{G}a}(\lambda_1) \cdot f_{\mathcal{G}a}(\lambda_2) \cdot f_N(\gamma_1) \cdot f_N(\gamma_2) \cdot f(\theta) \cdot L\left(\Delta Z_{ik}(t_j); \delta\right). \tag{9}$$

It can be noted that the joint posterior distribution in (9) results in a nonstandard complex form. However, the MCMC can be utilized to estimate the parameters of interest (δ) from model (9). The procedure consists in generating samples from this joint posterior distribution. In this case, the Gibbs sampling algorithm is utilized to obtain such samples from the joint distribution. Important information about this algorithm can be found in the works of Gelfand and Smith,[31] Casella and George,[32] Smith and Roberts,[33] and Gelman et al.[34] Generally, the algorithm consists in dividing the parameter vector into d subvectors, $\delta = (\delta_1, \ldots, \delta_d)$, such that each iteration of the algorithm cycles through the subvectors of δ, drawing each subset conditional on the value of all vectors. This process can be seen as generating a realization of a Markov chain that is built from a set of base transition probabilities. When the base transition probabilities are applied in sequence, the algorithm can be described as simulating a homogeneous Markov Chain $\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \ldots$, in such case, the procedure for generating $\delta^{(t)}$ from $\delta^{(t-1)}$ can be summarized as follows[35]:

$$\begin{aligned}
&\text{Pick } \delta_1^t \text{ from the distribution for } \delta_1 \text{ given } \delta_2^{(t-1)}, \delta_3^{(t-1)}, \ldots, \delta_n^{(t-1)}\\
&\text{Pick } \delta_2^t \text{ from the distribution for } \delta_2 \text{ given } \delta_1^{(t)}, \delta_3^{(t-1)}, \ldots, \delta_n^{(t-1)}\\
&\qquad\vdots\\
&\text{Pick } \delta_i^t \text{ from the distribution for } \delta_i \text{ given } \delta_1^{(t)}, \ldots, \delta_{i-1}^{(t)}, \delta_{i+1}^{(t-1)}, \ldots, \delta_n^{(t-1)}\\
&\qquad\vdots\\
&\text{Pick } \delta_n^t \text{ from the distribution for } \delta_n \text{ given } \delta_1^{(t)}, \delta_3^{(t)}, \ldots, \delta_{n-1}^{(t)}.
\end{aligned}$$

The new value for δ_{i-1} is used immediately when picking the next value for δ_i.

The implementation of the Gibbs sampling algorithm for the estimation of the parameters from Equation (9) is performed using the OpenBUGS package software.[36] Zeros trick is used in OpenBUGS given that the log-likelihood is not a standard distribution.[37] In general, the zeros trick consists in introducing the term $l(\delta; \Delta Z_{ik}(t_j))$ into the joint distribution. The trick consists in telling to OpenBUGS that a datum 0 has been observed from a Poisson distribution with mean $-l(\delta; \Delta Z_{ik}(t_j))$. If $-l(\delta; \Delta Z_{ik}(t_j))$ exceeds the unity, then $-l(\delta; \Delta Z_{ik}(t_j)) + Q$ is used for some suitable big Q. In Appendix A, the general code for the implementation of the zeros tricks under the different copula functions is provided.

4.1 | Performance comparison between the Bayesian approach and MLE

As previously stated, the estimation of parameters is performed via MCMC on the basis of the Gibbs sampling. However, another alternative may consist in obtaining the estimates of the parameters on the basis of MLE. To compare the performance of these 2 estimation methods, a simulation study is presented in this section. A bivariate IG model based on the

TABLE 2 Mean parameters estimations under MLE and Bayesian approach. MSE within parenthesis

Sample Size	Parameter	MLE	Bayesian
n = 30	μ_1	5.4141 (0.5195)	5.3941(0.2789)
	λ_1	7.8515 (6.8556)	7.574(5.9662)
	μ_2	6.3126 (0.4953)	6.2931(0.2132)
	λ_2	8.3749 (5.3325)	8.1789(4.9116)
	θ	10.8282 (1.2635)	10.7882(1.1569)
n = 50	μ_1	5.2705 (0.2631)	5.2144(0.1777)
	λ_1	7.7095 (1.7855)	7.5507(1.5241)
	μ_2	6.32467 (0.6637)	6.2572(0.4563)
	λ_2	8.2235 (1.3365)	8.0541(1.3159)
	θ	11.0136 (0.9853)	10.7867(0.9561)
n = 100	μ_1	5.1016 (0.0959)	5.1032(0.0649)
	λ_1	7.5853 (1.2978)	7.6137(1.1941)
	μ_2	6.1219 (0.1128)	6.1594(0.07588)
	λ_2	8.0909 (1.9297)	8.1032(2.0033)
	θ	10.1137 (0.9112)	10.1916(0.7588)

Frank copula is considered to perform the simulation study. Sample sizes of 30, 50, and 100 are simulated from the bivariate model with parameters ($\mu_1 = 5, \lambda_1 = 7, \mu_2 = 6, \lambda_2 = 8, \theta = 10$). By using the *copula*[38] package in R, the different samples were simulated and then fitted using the *fitMvdc* function to obtain the MLE estimations. The noninformative prior distributions described in the previous section are considered for $(\mu_1, \lambda_1, \mu_2, \lambda_2, \theta)$ in the Bayesian estimation scheme. By using OpenBUGS, MCMC chains of size 10 000 were taken for the parameters of interest. The average value of the estimates and their respective mean square error (MSE), based on 100 replications, are presented in Table 2.

From Table 2, it can be noted that the Bayesian estimators perform better than the obtained via MLE approach, but it can also be noted that the differences in estimations tend to be smaller as the sample size increases. Thus, it can be noted that the MSE tend to stabilize in the MLE approach as the sample size increases. It should be noted that, in some degradation analysis, there may be small sample sizes of data to perform the reliability assessment. In such situations, the Bayesian approach is a good option to perform the estimations of the parameters of interest.

5 | CASE STUDY

In this section, the case study is presented, which consisted in the fatigue-crack propagation of 2 cracks in 2 terminals of an electronic device. Each device has 2 terminals whose function is to transfer a signal to a receptor. These terminals are depicted in Figure 1A. As can be noted in Figures 1A and B, a wire is compressed between the 2 legs of the terminal. This is performed via the fusion welding process by 2 electrodes that make contact with the 2 legs of the terminal. Because of the mechanical stress caused by the electrodes, some cracks may be present in the terminals, as can be seen in Figure 1B.

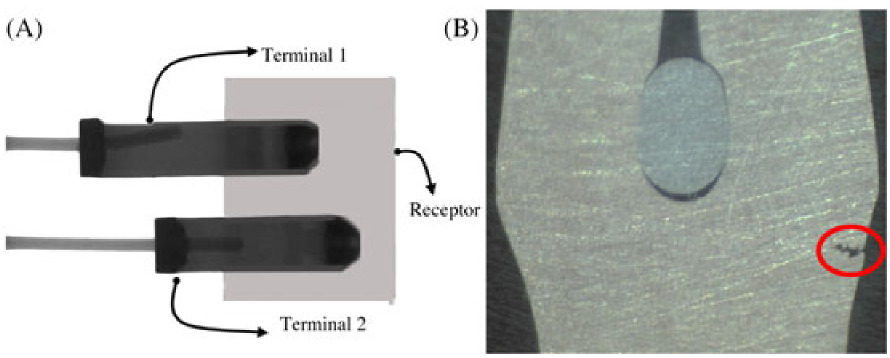

FIGURE 1 Configuration of terminals of the case study. A, terminals and receptor; B, crack of the terminal [Colour figure can be viewed at wileyonlinelibrary.com]

The propagation of the cracks to a certain critical length can lead to failure of the device because of the inability of transferring the signal to the receptor. Given that the fatigue-crack growth is of interest, a degradation test based on vibration fatigue can be considered to study the propagation of cracks in the terminals. Ten devices were available for the experimental study as every device has 2 terminals, 2 sets of fatigue-crack growth data sets were obtained. The 10 devices were mounted in a dynamic shaker device of a vibration chamber and were subjected to sinusoidal-type vibration profiles or cycles. The crack propagations for both terminals were measured every 0.1 hundred thousand cycle until 0.9 hundred thousand cycle. The measurements for every crack were performed at the same measurement times via a vision system. The obtained data is presented in Table 3, the units are in millimeters. In Figure 2, the crack degradation paths for the 2 terminals of every device are illustrated.

TABLE 3 Crack propagation data for terminals 1 and 2

		Hundred Thousands of Cycles									
Terminal	**Device**	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
	1	0	0.014	0.018	0.016	0.021	0.089	0.09	0.02	0.06	0.014
	2	0	0.031	0.017	0.075	0.011	0.024	0.025	0.08	0.01	0.043
	3	0	0.011	0.069	0.07	0.03	0.01	0.01	0.01	0.012	0.073
	4	0	0.03	0.02	0.08	0.03	0.05	0.06	0.09	0.02	0.055
Terminal 1	5	0	0.01	0.012	0.08	0.031	0.05	0.05	0.01	0.035	0.015
	6	0	0.011	0.05	0.09	0.026	0.084	0.085	0.022	0.036	0.016
	7	0	0.017	0.012	0.07	0.01	0.015	0.016	0.01	0.099	0.03
	8	0	0.026	0.016	0.01	0.01	0.012	0.01	0.01	0.021	0.016
	9	0	0.03	0.08	0.051	0.072	0.09	0.09	0.03	0.08	0.033
	10	0	0.08	0.012	0.016	0.032	0.01	0.01	0.02	0.013	0.034
	1	0	0.01	0.02	0.025	0.052	0.058	0.018	0.017	0.06	0.042
	2	0	0.09	0.071	0.011	0.075	0.012	0.022	0.09	0.03	0.028
	3	0	0.01	0.05	0.021	0.037	0.024	0.016	0.011	0.063	0.03
	4	0	0.016	0.06	0.011	0.017	0.023	0.071	0.01	0.01	0.04
Terminal 2	5	0	0.036	0.06	0.08	0.028	0.038	0.039	0.044	0.09	0.08
	6	0	0.014	0.088	0.01	0.082	0.083	0.012	0.016	0.03	0.056
	7	0	0.037	0.027	0.014	0.018	0.028	0.04	0.07	0.02	0.072
	8	0	0.035	0.051	0.019	0.069	0.093	0.01	0.07	0.014	0.023
	9	0	0.067	0.081	0.013	0.012	0.011	0.034	0.011	0.01	0.046
	10	0	0.025	0.027	0.012	0.012	0.075	0.036	0.018	0.017	0.04

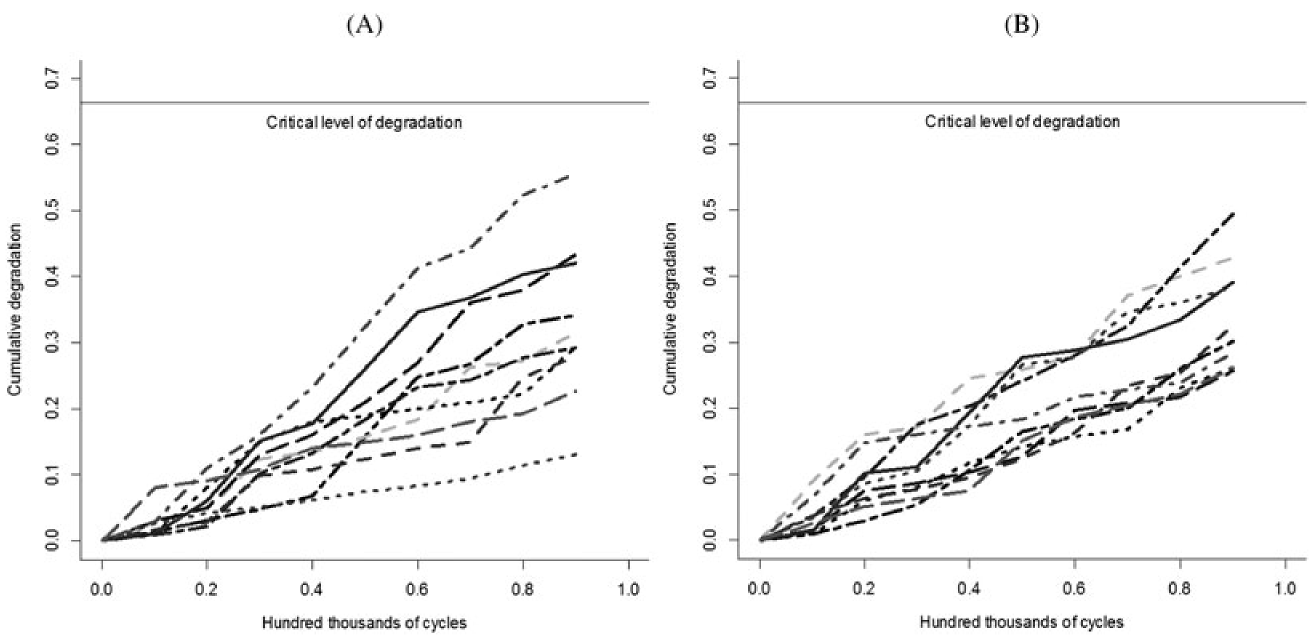

FIGURE 2 Cumulative degradation paths. A, Terminal 1; B, Terminal 2

The propagation of the cracks can be seen as a degradation process and, therefore, as a stochastic process. As the device has 2 terminals, the crack propagation of every terminal can be seen as degradation processes. Considering that the cracks are from different positions, it is important to assess the dependence of the crack propagation of the terminals for every device. It is considered that the device has failed if the length of any of the 2 cracks crosses the critical limit of 0.663 mm. Such critical level of degradation was obtained by considering the total width of the terminal defined by the customer of the product. In this case, if a crack length exceeds the total width of the terminal, a failure of the system can be obtained given the inability of the product to transfer a signal the receptor.

The fatigue-crack-growth data from the terminal 1 is considered to be modeled with an IG process $F_{\mathcal{IG}}\left(\Delta Z_{i1}(t_j)\right)$, whereas the fatigue-crack-growth data from the terminal 2 is also considered to be modeled with an IG process $F_{\mathcal{IG}}\left(\Delta Z_{i2}(t_j)\right)$. According to Ye and Chen[19] and Ye and Xie,[39] the IG process is a limiting compound Poisson process with independent and positive increments, which means that the degradation process can be considered as accumulations of additive and irreversible damage caused by a sequence of external random shocks. This certainly justifies the use of the IG process in the case study as can be noted in the behavior of the degradation paths in Figure 2. Furthermore, the first passage time distribution of the IG process is closely related to the Birnbaum-Saunders distribution,[19] which is a life distribution model that is derived from a physical fatigue process where crack growth causes a failure such as the case study presented in this paper.

The copula functions listed in Table 1 are used to describe the dependence of the IG processes in the form of $H(\Delta Z_{i1}(t_j), \Delta Z_{i2}(t_j))$. It can be noted from Figure 2, that the degradation paths are not linear in the function of the cycles; this justifies the use of the monotone time-scale transformation $\tau = t^{\gamma}$.

5.1 | Parameters estimation

The parameters of interest $\delta = (\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta)$ are estimated, in general, using MCMC via Gibbs the sampling by using a sample drawn from their respective posterior distributions as described in (9). The posterior distribution of parameters δ given the data is $P(\delta \mid Data)$, and is defined as $P(\delta \mid Data) \propto L(Data \mid \delta)\, P(\delta)$, where $L(Data \mid \delta)$ is the likelihood function of the model and $P(\delta)$ is the respective prior distribution for $\delta = (\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta)$. It is assumed that there is prior independence among the parameters. Estimation of the parameters was performed by using the algorithm presented in Appendix A and implemented in OpenBUGS. As the log-likelihood function in Equation (7) is not a standard distribution in OpenBUGS, the zeros trick was implemented in the algorithm specifying the log-likelihood function $l(\mu_1, \lambda_1, \gamma_1, \mu_2, \lambda_2, \gamma_2, \theta)$. Two sets of initial values are considered in the algorithm to assess the convergence of the parameters of interest; the sampling of the parameters was monitored using the analysis of the trace plots and the Brooks-Gelman-Rubin statistic.[40]

The prior distributions, as well as their hyperparameters, are given as follows. Noninformative a priori truncated normal distributions $TN\left(a_{\mu_k}, b_{\mu_k}\right)$ for parameters (μ_k) are considered. The hyperparameters are $a_k = 0$, for $k = 1, 2$, and precision parameter $1/b_k^2 = 1 \times 10^{-9}$, for $k = 1, 2$. For λ_k, the gamma hyperparameters are considered as $\vartheta_{\lambda_k} = 1$ and $\kappa_{\lambda_k} = 0.01$. For (γ_k, θ), noninformative a priori normal distributions were considered, with hyperparameters $a_{\gamma_k} = 0$ and precision parameter $1/b_{\gamma_k}^2 = 1\times10^{-9}$ and $a_\theta = 0$ and precision parameter $1/b_\theta^2 = 1\times10^{-9}$. A total of 5000 iterations were considered for burn-in, and 10 000 were considered for estimation purposes. A summary of the obtained estimations is presented in Table 4. As 2 sets of initial values were determined for every parameter, the Brooks-Gelman-Rubin statistic was calculated for the parameters of interest. In general, it was found that convergence is achieved in every parameter. In Appendix B, the trace plots for the estimated parameters of the bivariate models under different copulas are provided.

The estimation of the parameters of the IG processes considering nondependence were also performed by considering only the marginal IG distributions of both terminals without the copula functions; the results are presented in Table 5. In this case, the parameters were also estimated by using a Bayesian approach. Considering noninformative a priori gamma distributions $Ga\left(\vartheta_{\lambda_k}, \kappa_{\lambda_k}\right)$ for parameters (λ_k) and noninformative a priori truncated normal $\left(a_{\mu_k}, b_{\mu_k}\right)$ distributions for (μ_k).

From the credible intervals in Table 4, it can be noted that, in all bivariate models, the estimations of the parameters vary significantly, specifically in $(\mu_k, \lambda_k, \theta)$. Such significant variation can be reduced in all parameters by incorporating prior information about these parameters in the Bayesian estimation process. Considering the posterior parameters obtained in Table 4 as prior information and by integrating such information with the observed degradation data set in Table 3, a second implementation of the MCMC in OpenBUGS was carried out. The AHM and Frank copulas are considered to illustrate the informative process. It can be noted by the boxplots in Figure 3, that the variation in the estimation of the parameters of interest is reduced. This improvement of estimation results is mainly due to the evolutive integration of

TABLE 4 Summary of estimations obtained considering dependence between terminals

Copula	Parameter	Mean	Sd[a]	MC[b] Error	$t_{0.025}$	$t_{0.5}$	$t_{0.975}$
Plackett	γ_1	1.0390	0.0631	0.0003	0.9219	1.0360	1.1690
	γ_2	1.0270	0.0614	0.0003	0.9132	1.0250	1.1550
	λ_1	4.8460	0.7255	0.0079	3.5180	4.8080	6.3570
	λ_2	5.9750	0.9019	0.0093	4.3440	5.9200	7.9000
	μ_1	0.3789	0.0368	0.0009	0.3165	0.3759	0.4604
	μ_2	0.3855	0.0342	0.0006	0.3273	0.3827	0.4617
	θ	0.9095	0.2766	0.0033	0.4887	0.8681	1.5460
Frank	γ_1	1.0380	0.0631	0.0003	0.9214	1.0360	1.1690
	γ_2	1.0290	0.0634	0.0003	0.9118	1.0270	1.1590
	λ_1	4.8290	0.7248	0.0052	3.5200	4.7920	6.3640
	λ_2	5.9770	0.8906	0.0061	4.3550	5.9320	7.8710
	μ_1	0.3796	0.0379	0.0006	0.3162	0.3759	0.4639
	μ_2	0.3856	0.0346	0.0003	0.3266	0.3827	0.4610
	θ	−0.4427	0.5772	0.0145	−1.5700	−0.4355	0.6821
Gumbel	γ_1	1.0400	0.0628	0.0004	0.9220	1.0370	1.1710
	γ_2	1.0240	0.0607	0.0004	0.9123	1.0210	1.1490
	λ_1	4.8110	0.7147	0.0079	3.5270	4.7650	6.3220
	λ_2	5.9530	0.9015	0.0099	4.3190	5.9090	7.8270
	μ_1	0.3791	0.0376	0.0007	0.3148	0.3766	0.4595
	μ_2	0.3897	0.0350	0.0005	0.3305	0.3863	0.4669
	θ	1.0540	0.0436	0.0006	1.0020	1.0430	1.1610
Clayton	γ_1	1.0330	0.0632	0.0007	0.9169	1.0310	1.1630
	γ_2	1.0230	0.0627	0.0007	0.9068	1.0210	1.1540
	λ_1	4.7830	0.7217	0.0077	3.4820	4.7480	6.2940
	λ_2	5.9100	0.8829	0.0110	4.3060	5.8670	7.7590
	μ_1	0.3769	0.0374	0.0004	0.3135	0.3733	0.4629
	μ_2	0.3867	0.0353	0.0003	0.3262	0.3835	0.4659
	θ	0.0846	0.0749	0.0011	0.0023	0.0642	0.2736
Joe	γ_1	1.0420	0.0639	0.0006	0.9231	1.0390	1.1750
	γ_2	1.0250	0.0613	0.0005	0.9097	1.0230	1.1500
	λ_1	4.8100	0.7149	0.0074	3.5190	4.7730	6.3050
	λ_2	5.9750	0.8814	0.0095	4.3880	5.9340	7.8250
	μ_1	0.3826	0.0380	0.0005	0.3199	0.3783	0.4680
	μ_2	0.3921	0.0351	0.0003	0.3313	0.3892	0.4707
	θ	1.1020	0.0831	0.0014	1.0040	1.0820	1.3100
AMH	γ_1	1.0370	0.0626	0.0004	0.9234	1.0350	1.1670
	γ_2	1.0290	0.0638	0.0004	0.9120	1.0260	1.1620
	λ_1	4.8290	0.7178	0.0074	3.5200	4.7910	6.3400
	λ_2	5.9890	0.9111	0.0098	4.3560	5.9440	7.9150
	μ_1	0.3798	0.0379	0.0007	0.3151	0.3763	0.4664
	μ_2	0.3866	0.0351	0.0004	0.3278	0.3834	0.4640
	θ	−0.2330	0.2741	0.0029	−0.7909	−0.2241	0.2740

[a]Standard deviation
[b]Monte Carlo

TABLE 5 Summary of estimations obtained without considering dependence

Terminal	Parameter	Mean	Sd[a]	MC[b] Error	$t_{0.025}$	$t_{0.5}$	$t_{0.975}$
1	γ_1	1.038	0.06171	4.12×10^{-4}	0.9228	1.035	1.166
	λ_1	4.862	0.7274	0.003833	3.547	4.825	6.394
	μ_1	0.3776	0.03669	1.65×10^{-4}	0.3151	0.3741	0.4601
2	γ_2	1.026	0.0618	3.61×10^{-4}	0.912	1.024	1.155
	λ_2	6.013	0.8945	0.00464	4.382	5.969	7.9
	μ_2	0.3868	0.0341	1.41×10^{-4}	0.3287	0.384	0.461

[a]Standard deviation
[b]Monte Carlo

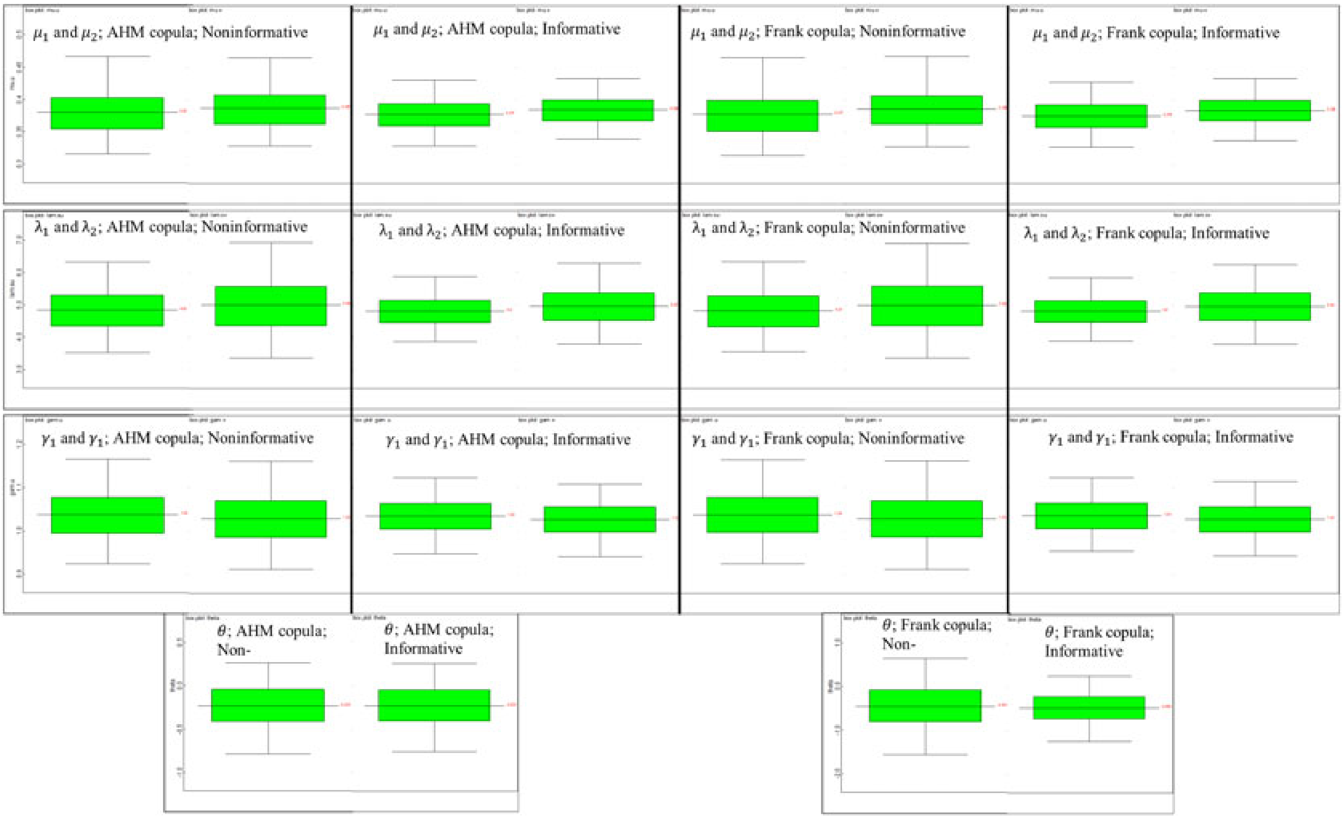

FIGURE 3 Boxplots comparison of noninformative and informative estimations for parameters of the bivariate AHM-IG and Frank-IG models [Colour figure can be viewed at wileyonlinelibrary.com]

posterior and prior distributions in the Bayesian estimation scheme. In addition, if historical information is available, it is expected that the estimation process performs even better. Such integration of information is critical for the situation when few degradation observations are available. Given that, as denoted in Section 4.1, the Bayesian scheme performs better than MLE when small sample sizes are considered, and MLE performs better as the sample size increases. Both aspects (few availability of degradation information and integration of prior information) present an advantage of the considered Bayesian estimation scheme for degradation analysis, which cannot be fulfilled with the MLE scheme.

Considering the estimates of the bivariate models obtained from the informative process, it is also possible to obtain information about the level of dependence between the degradation processes by considering the parameter θ from each copula function, which can be related to the Kendall coefficient in different forms depending on the treated copula.[41] The Kendall coefficient provides a good alternative to measure the level of dependence between the marginal distributions from the copula. Kendall's coefficients obtained from every copula are: 0.056 for Frank, 0.051 for Gumbel, 0.041 for Clayton, 0.055 for Joe, and −0.049 for AMH. In the case of the Plackett copula, the Kendall coefficient does not have a closed form (Fredricks and Nelsen, 2007), so the Spearman coefficient was computed as follows: 0.21.

In addition, information criteria can be used to obtain the best-fitting copula model. In order to obtain such model, the Akaike information criterion (AIC) is used, which is an important tool for model selection.[42] This criterion is defined as $AIC = -2 \times l(\delta) + 2k$, where, $l(\delta)$ is the evaluated log-likelihood function from Equation (7) for every copula, and k is the number of parameters. By using this criterion, the next results were obtained: −8213 for Plackett, −8296 for Frank, −8216 for Gumbel, −8210 for Clayton, −8204 for Joe, and −8213 for AMH. As can be noted, there are slightly differences for the AIC of every copula function. However, the Frank copula has the smallest value; this may indicate that the Frank copula is the better-fitting model for the marginal degradation processes and their dependence structure. However, the Kendall coefficient from the Frank copula is 0.056, which indicates nondependence; as when the Kendall coefficient from the Frank copula tends to 0, it implies nondependence. In Figure 4, the bivariate PDFs of the best-fitting copulas are illustrated.

We also considered the goodness of fit testing for the bivariate models based on copula functions. The test is based on an empirical process that compares the empirical copula with a parametric estimate of the copula derived under the null hypothesis $H_0 : C \in C_0$, which indicates that the dependence structure of the multivariate copula based distribution (C)

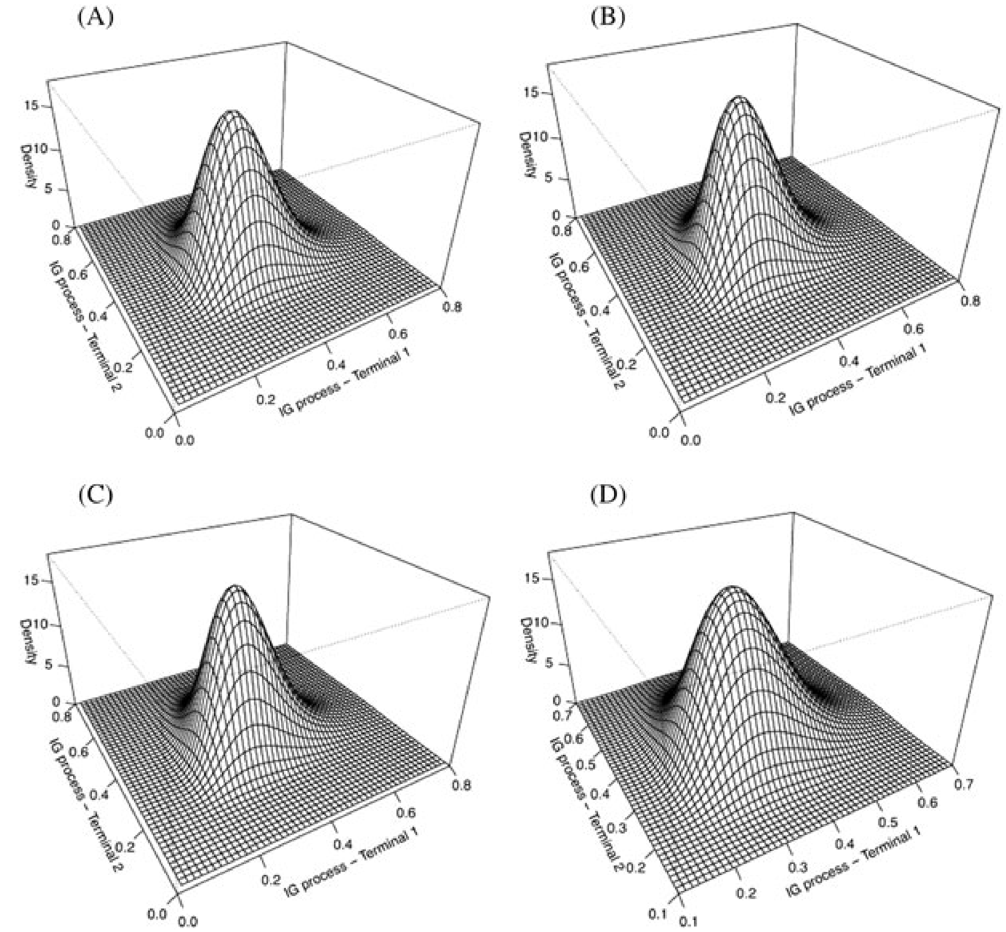

FIGURE 4 Bivariate PDF for copula functions: A, Frank; B, Gumbel; C, Clayton; D, Joe

is well represented by a specific parametric family (C_0) of copulas. The test statistic is based on the rank-based version of the Cramer-von Mises statistic.[43] Large values of this statistic lead to the rejection of the null hypothesis. In addition, p-values were computed using parametric bootstrap by considering the copula package[38] in the R software. The results were obtained as follows for the different copulas: for the placket copula, the test statistic and p-value were obtained as 0.07575 and 0.03178, respectively; for the Frank copula, 0.02452 and 0.0515, respectively; for the Gumbel copula, 0.039361 and 0.03731; for the Clayton copula, 0.098661 and 0.02488; for the Joe copula, 0.11833 and 0.007463; for the AMH copula, 0.07976 and 0.007463. It can be noted that, if a significance level of 0.05 is considered, then the Frank copula is the only not rejected, which confirms the best-fitting copula selected via the information criterion.

5.2 | Comparison with gamma and geometric Brownian motion stochastic processes

As mentioned before, there are some other alternative models that may be considered to model the degradation data in Table 3. The gamma and geometric Brownian motion (GBM) processes are good options. In this section, the degradation data set is fitted to such stochastic processes considering that the best fitting copula is the Frank copula. Considering that, during the degradation test, N units are tested and M measurements for all the units are observed up to the termination time T, which results in degradation measurements $Z_{ik}(t_j)$ of the ith unit at the corresponding time t_j, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$ and $k = 1, 2$ PC, and the time-scale transformation $\tau(t) = \tau(t, \gamma)$, thus the corresponding PDF of the gamma and GBM motion processes are defined as follows, respectively:

$$f_{\mathcal{G}a}\left(\Delta Z_i(t_j)\right) = \frac{\Delta Z_i(t_j)^{\nu\Delta\tau(t_j,\gamma)-1}}{\Gamma\left(\nu\Delta\tau(t_j,\gamma)\right) u^{\nu\Delta\tau(t_j,\gamma)}} \exp\left\{-\frac{\Delta Z(t_j)}{u}\right\}$$

$$f_{\mathrm{GBM}}\left(\Delta Z_i(t_j)\right) = \frac{1}{\sqrt{2\pi\Delta\tau(t_j,\gamma)}\sigma'\Delta Z_i(t_j)} \exp\left\{-\frac{\left(\ln\left(\Delta Z_i(t_j)\right) - \alpha'\Delta\tau(t_j,\gamma)\right)^2}{2\sigma'^2\Delta\tau(t_j,\gamma)}\right\}.$$

Thus, the corresponding bivariate models based on copula functions for the gamma and GBM processes are defined as follows, for any of the 2 stochastic processes:

$$f\left(Z_{i1}(t_j), Z_{i2}(t_j)\right) = c\left(F_1\left(Z_{i1}(t_j)\right), F_2\left(Z_{i2}(t_j)\right)\right) \times f_1\left(Z_{i1}(t_j)\right) \times f_2\left(Z_{i2}(t_j)\right)$$

with $k = 1, 2$ PC, $i = 1, 2, \ldots, N, j = 1, 2, \ldots, M$.

The log-likelihood function is described as in (7) as

$$l(\delta) = \sum_{i=1}^{N}\sum_{j=1}^{M}\left[\ln\left(C\left(U_{ij1}, V_{ij2}; \theta\right)\right) + \sum_{k=1}^{2}\ln\left(f_{\mathcal{G}a \text{ or GBM}}\left(\Delta Z_{ik}(t_j)\right)\right)\right].$$

Information criteria are also considered to define the best-fitting model. The gamma and GBM bivariate models were fitted to the degradation data set in Table 3, and the corresponding AIC values were obtained. For the bivariate IG based on the Frank copula model, the AIC value was observed as −8296, the AIC value obtained for the bivariate gamma based on the Frank copula model was −8418, whereas the AIC value obtained for the bivariate GBM based on the Frank copula model was −3230. As the bivariate model based on IG processes has the lowest value of AIC, it is considered that the bivariate IG model is the best-fitting model for the presented degradation data set as the marginal PDF in the bivariate copula function can be defined before the bivariate joint model is fitted. The degradation data sets for terminals 1 and 2 were fitted to the IG, gamma, and GBM stochastic processes marginally. The AIC criterion was also considered with the next results: for terminal 1, the AIC value for the IG process resulted in −4190, for the gamma process in −4160, and for the GBM process in −4034. It can be noted that the IG process has the lowest value of AIC, so the IG process is the best-fitting model for terminal 1. For terminal 2, the AIC value for the AIC resulted in −4162, for the gamma process in −4097, and for the GBM process in −4022. It can also be noted that the IG process has the lowest value of AIC, so the IG process is also the best-fitting model for terminal 2.

5.3 | Reliability estimation

Considering the estimated parameters of the bivariate IG-Frank model, it is possible to assess the reliability of the device under study taking into account the Equations (3) and (8). In addition, taking into account the estimates in Table 5, it is possible to assess the reliability of the device considering that there is no dependence between the degradation processes. The reliability functions are compared in Figures 5 and 6 for terminals 1 and 2, respectively.

The nondependence assessment can be confirmed by the reliability functions in Figures 5 and 6, where it can be noted that the functions from the dependent and independent scenarios constantly overlap from the beginning to

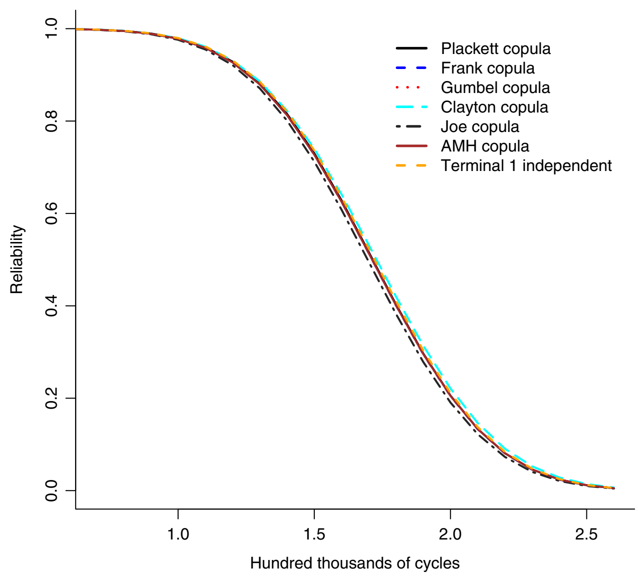

FIGURE 5 Comparison of reliability functions with and without dependency for terminal 1 [Colour figure can be viewed at wileyonlinelibrary.com]

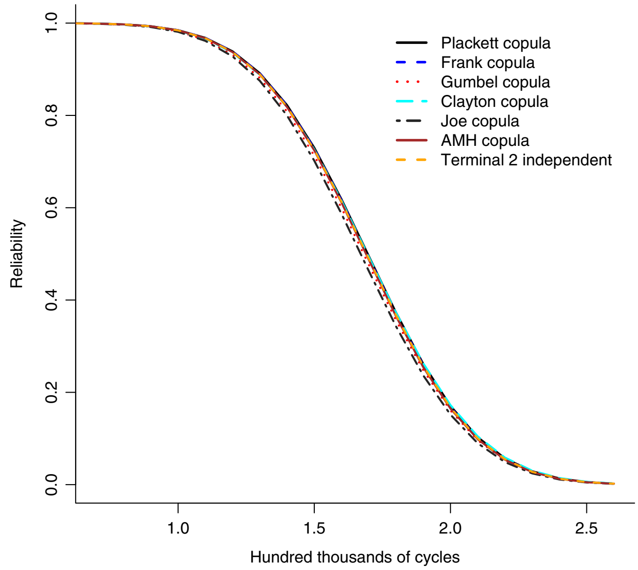

FIGURE 6 Comparison of reliability functions with and without dependency for terminal 2 [Colour figure can be viewed at wileyonlinelibrary.com]

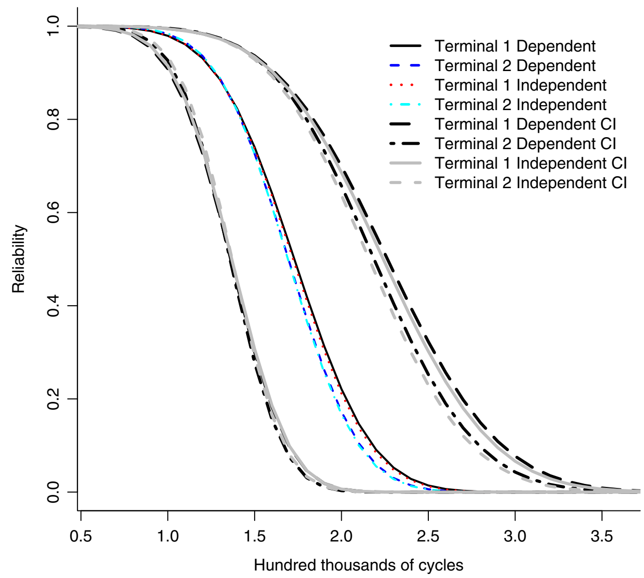

FIGURE 7 Comparison of reliability functions for the best-fitting dependent copula *y* nondependent models [Colour figure can be viewed at wileyonlinelibrary.com]

the end. In addition, in Figure 7, the reliability functions, along with the corresponding credible intervals, of the 2 terminals considering the best-fitting copula, are presented. The reliability functions of the terminals considering the obtained estimations under the independent scenario are also presented in Figure 7. It can be noted that, as in Figures 5 and 6, the reliability functions, along with credible intervals in Figure 7, are almost identical. Thus, the 2 terminals may be deemed as independent, and the reliability assessment may be obtained individually without considering the dependence structure. In addition, all the reliability functions of terminal 2 are smaller most of the time than the reliability functions of terminal 1. Even if there is no dependence between the degradation processes of the terminals, both are parallel components of the electronic device, and the failure of either can cause a failure of the device. Based on this, it can be noted that the reliability of the product should be decided by terminal that first reaches the critical level of degradation, which, in this case, may be the terminal 2.

6 | CONCLUDING REMARKS AND DISCUSSION

This article proposes a joint bivariate modeling of 2 degradation processes governed by IG processes via copula functions; the estimation of the parameters of interest is performed with MCMC via the Gibbs sampling, and reliability insights are provided to deal with the dependent and independent degradation of 2 PC. From the case study presented, it was found that the dependence structure between the degradation processes can be assessed by the copula functions. In this case, the best-fitting copula is the Frank copula; nevertheless, the Kendall coefficient is close to 0, which may indicate that there is no dependence between the degradation processes. It can be noted from Figure 7, that the reliability functions of the dependent and independent scenarios constantly overlap, which denotes that the degradation of the 2 terminals are indeed independent. Thus, the reliability should be assessed independently. The validity of the IG process was also checked by fitting the degradation data set to the gamma and geometric Brownian motion processes. It was found that, according to the AIC values, the IG process fits better the data set either jointly and marginally. In addition, as mentioned in the introduction section, the Bayesian estimation approach has the advantages of a good performance when few degradation measures are available and the possibility of incorporating prior information in the modeling. Such advantages have been cleared by performing, in first instance, a simulation study considering the MCMC-based Gibbs sampling estimation scheme and the MLE and comparing the performances via MSE. It was found that, with small sample sizes, the Bayesian estimation scheme performs better than the MLE. Furthermore, the estimations perform even better by introducing prior information as denoted in Figure 3. The case study presented may be used for further research. It can be noted that none of the terminals reached the critical level of degradation. Given that the reliability function depends on the degradation trajectories reaching such critical value, further research is needed. However, it should be noted that the reliability function of the first passage time distribution of the IG process is closed in the form of an IG distribution. Thus, the estimated parameters of the IG process directly define the first passage time distribution in terms of the critical level of degradation and the estimated parameters of the IG process, as can be noted in (4). Further research may be directed to consider the confidence intervals of the estimated IG process parameters and performing Monte Carlo simulations with (4), such that a wider variety of parameters estimations in terms of the first passage time distribution can be assessed. In addition, future work can be extended considering the characteristics of the case study; this could be treated as an s-dependent competing risk model.

ORCID

Luis Alberto Rodríguez-Picón http://orcid.org/0000-0003-2951-2344

REFERENCES

1. Lu CJ, Meeker WQ. Using degradation measures to estimate a time-to-failure distribution. *Technometrics*. 1993;35(2):161-174.
2. Meeker WQ, Escobar LA. *Statistical Methods for Reliability Data*. New York, NY: John Wiley & Sons Inc; 1998.
3. Lawless J, Crowder M. Covariates and random effects in a gamma process model with application to degradation and failure. *Lifetime Data Anal*. 2004;10(3):213-227.
4. Ye Z-S, Xie M, Tang L-C, Shen Y. Degradation-based burn-in planning under competing risks. *Technometrics*. 2012;54(2):159-168.
5. Guida M, Postiglione F, Pulcini G. A time-discrete extended gamma process for time-dependent degradation phenomena. *Reliab Eng Syst Saf*. 2012;105:73-79.
6. Ye Z-S, Xie M, Tang L-C, Chen N. Semiparametric estimation of gamma processes for deteriorating products. *Technometrics*. 2014;56(4):504-513.
7. Bagdonavicius V, Nikulin MS. Estimation in degradation models with explanatory variables. *Lifetime Data Anal*. 2000;7(1):85-103.
8. Park C, Padgett WJ. Accelerated degradation models for failure based on geometric Brownian motion and gamma processes. *Lifetime Data Anal*. 2005;11(4):511-527.
9. Park C, Padgett WJ. Stochastic degradation models with several accelerating variables. *IEEE Trans Reliab*. 2006;55(2):379-390.
10. van Noortwijk JM. A survey of the application of gamma processes in maintenance. *Reliab Eng Syst Saf*. 2009;94(1):2-21.
11. Bordes L, Paroissin C, Salami A. Parametric inference in a perturbed gamma degradation process. 2010. arXiv:1005.1214v1.
12. Pan Z, Balakrishnan N. Reliability modeling of degradation of products with multiple performance characteristics based on gamma processes. *Reliab Eng Syst Saf*. 2011;96(8):949-957.
13. Rodríguez-Picón LA, Méndez-González LC, Borbón MIR, Del Valle A. Reliability estimation for products subjected to two-stage degradation tests based on a gamma convolution. *Qual Reliab Eng Int*. 2016;32(8):2901-2908.
14. Pan Z, Feng J, Sun Q. Lifetime distribution and associated inference of systems with multiple degradation measurements based on gamma processes. *Eksploatacja i Niezawodnosc Maint Reliab*. 2016;18(2):307-313.

15. Pan Z, Sun Q, Feng J. Reliability modeling of systems with two dependent degrading components based on gamma processes. *Commun Stat Theory Methods*. 2016;45(7):1923-1938.
16. Elsayed EA, Liao HT. A geometric Brownian motion model for field degradation data. *Int J Mater Prod Technol*. 2004;20(1-3):51-72.
17. Park C, Padgett WJ. New cumulative damage models for failure using stochastic processes as initial damage. *IEEE Transactions on Reliab*. 2005b;54(3):530-540.
18. Chiang J-Y, Lio YL, Tsai T-R. Degradation tests using geometric Brownian motion process for lumen degradation data. *Qual Reliab Eng Int*. 2015;31(8):1797-1806.
19. Ye Z-S, Chen N. The inverse Gaussian process as a degradation model. *Technometrics*. 2014;56(3):302-311.
20. Wang X, Xu D. An inverse Gaussian process model for degradation data. *Technometrics*. 2010;52(2):188-197.
21. Peng CY. Inverse Gaussian processes with random effects and explanatory variables for degradation data. *Technometrics*. 2014;57(1):100-111.
22. Liu Z, Ma X, Yang J, Zhao Y. Reliability modeling for systems with multiple degradation processes using inverse Gaussian process and copulas. *Math Probl Eng*. 2014;2014(1-2):1-10. https://doi.org/10.1155/2014/829597
23. Sari JK, Newby MJ, Brombacher AC, Tang LC. Bivariate constant stress degradation model: LED lightning system reliability estimation with two-stage modelling. *Qual Reliab Eng Int*. 2009;25(8):1067-1084.
24. Pan Z, Balakrishnan N, Sun Q, Zhou J. Bivariate degradation analysis of products based on Wiener processes and copulas. *J Stat Comput Simul*. 2013;83(7):1316-1329.
25. Wang Y, Pham H. Modeling the dependent competing risks with multiple degradation processes and random shock using time-varying copulas. *IEEE Trans Reliab*. 2012;61(1):13-22.
26. Peng W, Li Y-F, Yang Y-J, Huang H-Z, Zuo MJ. Inverse Gaussian process models for degradation analysis: a Bayesian perspective. *Reliab Eng Syst Saf*. 2014;130:175-189.
27. Whitmore GA, Schenkelberg F. Modeling accelerated degradation data using Wiener diffusion with a time scale transformation. *Lifetime Data Anal*. 1997;3(1):27-45.
28. Nelsen RB. *An Introduction to Copulas*. 2nd ed. New York, NY: Springer; 2006.
29. Sklar A. Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de L'Université de Paris*. 1959;8:229-231.
30. Genest C, Favre A-C. Everything you always wanted to know about copula modeling but were afraid to ask. *J Hydrol Eng*. 2007;12(4):347-368.
31. Gelfand AE, Smith AFM. Sampling-based approaches to calculating marginal densities. *J Am Stat Assoc*. 1990;85(410):398-409.
32. Casella G, George EI. Explaining the Gibbs sampler. *Am Stat*. 1992;46(3):167-174.
33. Smith AFM, Roberts GO. Bayesian computation via the Gibbs sampler and related Markov chain Monte Carlo methods. *J R Stat Soc Ser B*. 1993;55(1):3-24.
34. Gelman A, Carlin JB, Stern HS, Rubin DB. *Bayesian Data Analysis*. 2nd ed. Boca Raton, FL: Chapman & Hall/CRC; 2009.
35. Neal R. Probabilistic inference using Markov chain Monte Carlo methods [Technical Report CRG-TR-93-1]. Toronto, Canada: University of Toronto; 1993. http://www.cs.toronto.edu/pub/radford/review.pdf
36. Lunn D, Spiegelhalter D, Thomas A. Best N. The BUGS project: evolution, critique and future directions. *Stat Med*. 2009;28(25):3049-3082.
37. Ntzoufras I. *Bayesian Modeling Using WinBUGS*. Hoboken, NJ: John Wiley & Sons, Inc.; 2009.
38. Yan J. Enjoy the joy of copulas: with a package copula. *J Stat Softw* 2007; 21(4):1-21. https://doi.org/10.18637/jss.v021.i04
39. Ye Z-S, Xie M. Stochastic modelling and analysis of degradation for highly reliable products. *Appl Stochastic Models Bus Ind*. 2015;31(1):16-32.
40. Gelman A, Rubin DB. Inference from iterative simulation using multiple sequences. *Stat Sci*. 1992;7(4):457-511.
41. Fredricks GA, Nelsen RB. On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables. *J Stat Plan Infer*. 2007;137(7):2143-2150.
42. Akaike H. A new look at the statistical model identification. *IEEE Trans Autom Control*. 1974;19(6):716-723.
43. Genest C, Rémillard B, Beaudoin D. Goodness-of-fit tests for copulas: a review and a power study. *Insur Math Econ*. 2009;44(2):199-213.

How to cite this article: Rodríguez-Picón LA, Rodríguez-Picón AP, Alvarado-Iniesta A. Degradation modeling of 2 fatigue-crack growth characteristics based on inverse Gaussian processes: A case study. *Appl Stochastic Models Bus Ind*. 2019;35:504–521. https://doi.org/10.1002/asmb.2329

APPENDIX A: OpenBUGS CODE FOR ESTIMATION OF THE BIVARIATE MODELS UNDER DIFFERENT COPULA FUNCTIONS

```
model {
C <- 1000000
for (i in 1:N)
{
for(j in 1:M-1)
{
zeros[i,j] <- 0
zeros[i,j] ~ dpois(zeros.mean[i,j])
zeros.mean[i,j] <- -L[i,j] + C
# Frank copula function
#L[i,j] <- log( theta * exp( -theta * ( u[i,j] + v[i,j] ) ) * ( 1 - exp( -theta ) ) / ( exp( -theta ) + exp( -theta *
#( u[i,j] + v[i,j] ) ) - exp( -theta * u[i,j] ) - exp( -theta * v[i,j] ) ) / ( exp( -theta ) + exp( -theta * ( u[i,j] + v[i, j] ) )
#- exp( -theta* u[i,j] ) - exp( -theta * v[i,j] )))
# Joe copula function
#L[i,j] <- log( pow( ( pow( (1 - u[i,j]), theta) + pow( (1 - v[i,j]), theta) - (pow( (1 - u[i,j]), theta) * pow( (1 - v[i,j]),
#theta) ) ), ((1/theta) - 2)) * pow( (1 - u[i,j]), (theta - 1)) * pow( (1 - v[i,j]), (theta - 1)) * ( theta - 1 + pow( (1 - u[i,j]),
#theta) + pow( (1 - v[i,j]), theta) - pow( (1 - u[i,j]), theta) * pow( (1 - v[i,j]), theta)))
# Clayton copula function
L[i,j] <- log( pow( pow( u[i,j], - theta) + pow( v[i,j], - theta) - 1, -1 / theta - 2 ) * pow( u[i,j], - theta -1) * pow( v[i,j],
- theta - 1) * (1 + theta))
# Placket copula function
#L[i,j] <- log( (theta * (1 + ((u[i,j] - (2 * u[i,j] * v[i,j]) ) + v[i,j]) * (theta - 1))) / pow( (pow((1 + (theta - 1) * (u[i,j] + v[i,j])),
2) - 4 * u[i,j] * v[i,j] * theta * (theta - 1)), 3/2))
# AMH copula function
#L[i,j] <- log( (1 + (theta * ( (1 + u[i,j]) * (1 + v[i,j]) - 3)) + (pow(theta, 2) * (1 - u[i,j]) * (1 - v[i,j]) )) / pow( (1 - (theta * (1
- u[i,j]) * (1 - v[i,j]))), 3))
# Gumbel copula function
#L[i, j] <- log( exp( -pow( pow( -log(u[i,j]) ,theta) + pow( -log(v[i,j]), theta) , 1/theta) ) * pow(-log(u[i,j]), theta-1) *
#pow( -log(v[i,j]), theta-1) / u[i,j] / v[i,j] * pow( pow( -log(u[i,j]), theta) + pow( -log(v[i,j]), theta),1/theta - 2) *
#( pow( pow( -log(u[i,j]), theta) + pow( -log(v[i,j]), theta), 1/theta) + theta -1 ))

#Cumulative distributions for inverse Gaussian processes
u[i,j]<- phi(sqrt(lam.su / Z.u[i,j]) * ((Z.u[i,j] / mu.u) - ts.u[i,j] )) + (exp((2*lam.u[i,j]) / miu.u[i,j]) * phi(-sqrt(lam.su /
Z.u[i,j]) * (ts.u[i,j] + (Z.u[i,j] / mu.u))))
v[i,j]<- phi(sqrt(lam.sv / Z.v[i,j]) * ((Z.v[i,j] / mu.v) - ts.v[i,j] )) + (exp((2*lam.v[i,j]) / miu.v[i,j]) * phi(-sqrt(lam.sv / Z.v[i,j])
* (ts.v[i,j] + (Z.v[i,j] / mu.v))))
# Marginal inverse Gaussian distributions
Z.u[i,j] ~ dinv.gauss(miu.u[i,j], lam.u[i,j])
Z.v[i,j] ~ dinv.gauss(miu.v[i,j], lam.v[i,j])
#Inverse Gaussian parameters with time scale transformation
miu.v[i,j] <- mu.v * ts.v[i,j]
lam.v[i,j] <- lam.sv * (pow(ts.v[i,j], 2))
ts.v[i,j]<-(pow( t[j+1] ,gam.v) - pow( t[j], gam.v ))
#Inverse Gaussian parameters with time-scale transformation
miu.u[i,j] <- mu.u * ts.u[i,j]
lam.u[i,j] <- lam.su * (pow(ts.u[i,j], 2))
ts.u[i,j]<- (pow( t[j+1] ,gam.u) - pow( t[j], gam.u ))
}
}
#theta ~ dnorm(0, 1.0E-9)        #a priori distribution for Frank copula function
```

```
#theta ˜ dunif(1, 100)            #a priori distribution for Gumbel copula function and Joe copula function
theta ˜ dunif(1.0E-6, 100)        #a priori distribution for Clayton copula function and Plackett copula function
#theta ˜ dunif(-1,1)              #a priori distribution for AMH copula function
mu.u ˜ dnorm(0, 1.0E-9) I(0, )    #a priori distribution for mean parameter of inverse Gaussian process
gam.u ˜ dnorm(0, 1.0E-9)          #a priori distribution for time-scale transformation parameter
lam.su ˜ dgamma(1, 0.01)          #a priori for shape parameter of inverse Gaussian process
mu.v ˜ dnorm(0, 1.0E-9) I(0, )    #a priori distribution for mean parameter of inverse Gaussian process
gam.v ˜ dnorm(0, 1.0E-9)          #a priori distribution for time-scale transformation parameter
lam.sv ˜ dgamma(1, 0.01)          #a priori for shape parameter of inverse gaussian process
}
list(t=c(0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9), N=10, M=10,
Z.u= structure(
.Data = c(
0.014,  0.018,  0.016,  0.021,  0.089,  0.090,  0.020,  0.060,  0.014,
0.031,  0.017,  0.075,  0.011,  0.024,  0.025,  0.080,  0.010,  0.043,
0.011,  0.069,  0.070,  0.030,  0.010,  0.010,  0.010,  0.012,  0.073,
0.030,  0.020,  0.080,  0.030,  0.050,  0.060,  0.090,  0.020,  0.055,
0.010,  0.012,  0.080,  0.031,  0.050,  0.050,  0.010,  0.035,  0.015,
0.011,  0.050,  0.090,  0.026,  0.084,  0.085,  0.022,  0.036,  0.016,
0.017,  0.012,  0.070,  0.010,  0.015,  0.016,  0.010,  0.099,  0.030,
0.026,  0.016,  0.010,  0.010,  0.012,  0.010,  0.010,  0.021,  0.016,
0.030,  0.080,  0.051,  0.072,  0.090,  0.090,  0.030,  0.080,  0.033,
0.080,  0.012,  0.016,  0.032,  0.010,  0.010,  0.020,  0.013,  0.034),
.Dim = c(10, 9)),
Z.v= structure(
.Data = c(
0.010,  0.020,  0.025,  0.052,  0.058,  0.018,  0.017,  0.060,  0.042,
0.090,  0.071,  0.011,  0.075,  0.012,  0.022,  0.090,  0.030,  0.028,
0.010,  0.050,  0.021,  0.037,  0.024,  0.016,  0.011,  0.063,  0.030,
0.016,  0.060,  0.011,  0.017,  0.023,  0.071,  0.010,  0.010,  0.040,
0.036,  0.060,  0.080,  0.028,  0.038,  0.039,  0.044,  0.090,  0.080,
0.014,  0.088,  0.010,  0.082,  0.083,  0.012,  0.016,  0.030,  0.056,
0.037,  0.027,  0.014,  0.018,  0.028,  0.040,  0.070,  0.020,  0.072,
0.035,  0.051,  0.019,  0.069,  0.093,  0.010,  0.070,  0.014,  0.023,
0.067,  0.081,  0.013,  0.012,  0.011,  0.034,  0.011,  0.010,  0.046,
0.025,  0.027,  0.012,  0.012,  0.075,  0.036,  0.018,  0.017,  0.040),
.Dim = c(10, 9)))
#Initials for Frank copula
list(gam.u = 1.5, lam.su = 10, mu.u = 10, gam.v = 1.5, lam.sv = 10, mu.v = 10, theta = 4)
list(gam.u = 2.5, lam.su = 15, mu.u = 5, gam.v = 2.5, lam.sv = 15, mu.v = 5, theta = 2)
#Initials for Gumbel and Joe copula
list(gam.u = 2.5, lam.su = 15, mu.u = 15, gam.v = 2.5, lam.sv = 15, mu.v = 15, theta = 2)
list(gam.u = 1.5, lam.su = 5, mu.u = 0.5, gam.v = 1.5, lam.sv = 5, mu.v = 0.5, theta = 4)
#Initials for Clayton and Plackett copula
list(gam.u = 1.5, lam.su = 10, mu.u = 1, gam.v = 1.5, lam.sv = 10, mu.v = 2, theta = 4)
#Initials for AMH copula
list(gam.u = 2, lam.su = 15, mu.u = 1.5, gam.v = 2, lam.sv = 25, mu.v = 2, theta = 0.5)
```

APPENDIX B: TRACE PLOTS OF THE ESTIMATED PARAMETERS FOR THE BIVARIATE MODELS UNDER DIFFERENT COPULAS

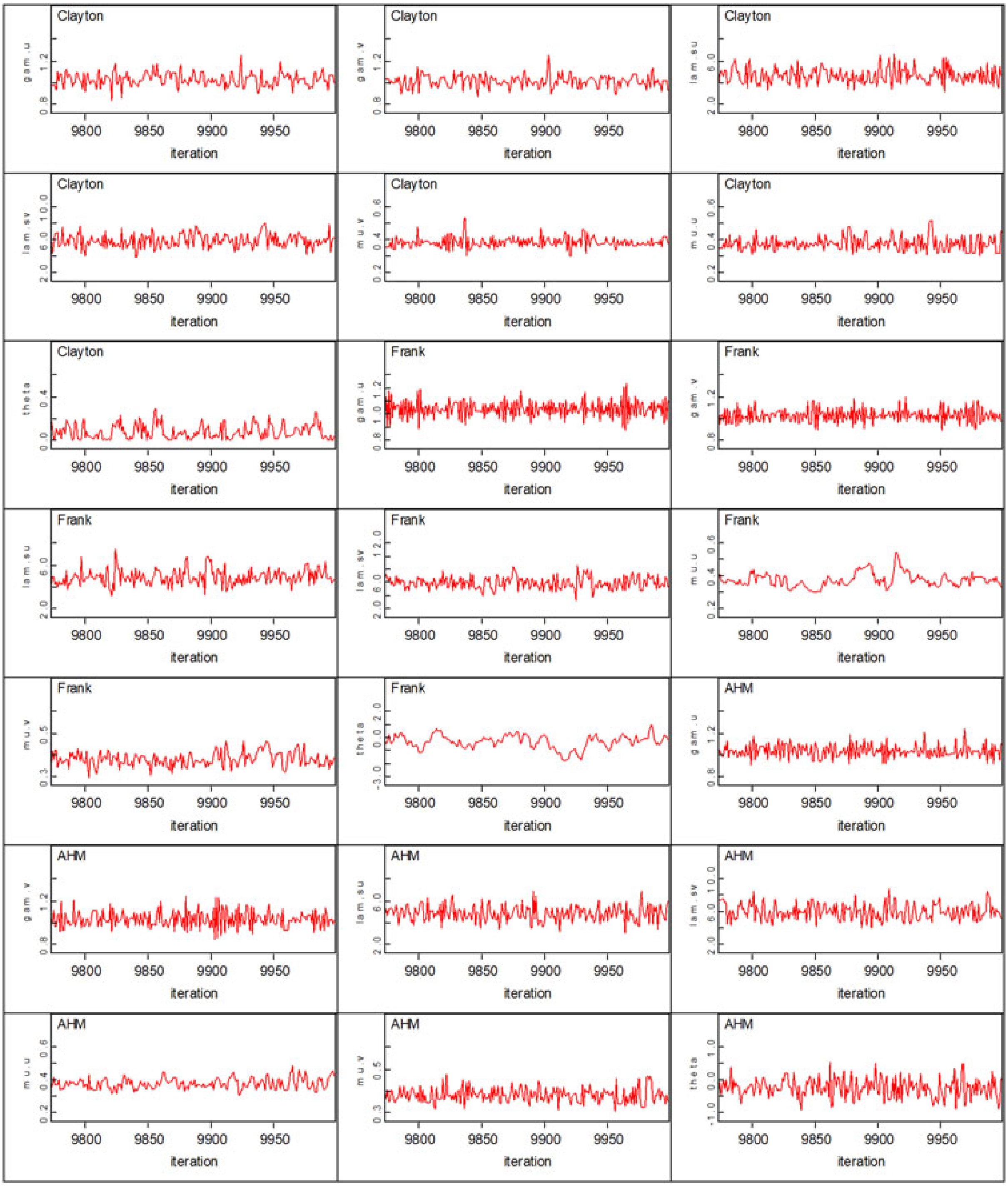

FIGURE B1 Trace plots of the estimated parameters for the bivariate models under several copulas [Colour figure can be viewed at wileyonlinelibrary.com]