Zariski topology for multiplication modules with applications to frames, quantales and classical Krull dimension

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Abstract

For a multiplication R-module M we consider the Zariski topology in the set $Spec\,(M)$ of prime submodules of M. We investigate the relationship between the algebraic properties of the submodules of M and the topological properties of some subspaces of $Spec\,(M)$. We also consider some topological aspects of certain frames. We prove that if R is a commutative ring and M is a multiplication R-module, then the lattice $Semp\,(M/N)$ of semiprime submodules of M/N is a spatial frame for every submodule N of M. When M is a quasi projective module, we obtain that the interval $[N, M] = \{P \in Semp\,(M) \mid N \subseteq P\}$ and the lattice $Semp\,(M/N)$ are isomorphic as frames. Finally, as applications we obtain results about quantales and the classical Krull dimension of M.

Introduction

Multiplication modules were introduced by Barnard [5], these modules have been studied by several authors [2], [3], [12], [18], [22] and [24] . The relationship between the algebraic properties of a ring and the topological properties of the Zariski topology defined on its prime spectrum has been studied in [13], [14], [15], [21][28]. Some notions of primeness have been introduced and investigated in [11], [25], [26]. In this paper, we consider the concept of prime and semiprime modules given in [19] , [20]. Given a multiplication module M over a commutative ring R, we consider the Zariski topology for the spectrum $Spec\,(M)$ of prime submodules of M. Motived by the results of the Zariski topology [4], [8], we investigate the relationship between the topological properties of some subspaces of $Spec\,(M)$ and the algebraic properties of the submodules of M.

In [16], [17] the authors introduce a framework of a lattice structure theory to analyze the submodules of a given module; in particular, they specialize in the lattice $Sub\,(M)$ of submodules of M and they obtain interesting results. These authors also observe some topological

aspects of certain frames that were constructed in that paper and that consideration eventually leads to the construction of some spatial frames. A spatial frame F is a frame which is a lattice isomorphic to the set of open subsets of topological space X. In this paper we take that point of view and give some interesting results for multiplication modules.

The organization of the paper is as follows: Section 1 provides the necessary material that is needed for the reading of the next sections. Section 2 is dedicated to prime (semiprime) modules. We give the relationship between prime (semiprime) submodules of a multiplication R-module M and prime (semiprime) ideals of the ring R. In Section 3 we consider the Zariski Topology for a multiplication module M and we study open and closed sets. Section 4 is dedicated to studying compact, irreducible and dense subspaces. We characterize compact sets in the form $\mathcal{U}(N)$ in terms of finitely generated submodules of M. In section 5 we give the main results and applications. We prove that $\{Semp\,(M)\,,\,\wedge\,,\,\vee\}$ is a frame for every ring R and every multiplication R-module M. We also prove that if R is a commutative ring and M is a multiplication R-module, then $Sub\,(M)$ is a bilateral quantal. Moreover, we prove that $Semp\,(M/N)$ is a spatial frame for all submodules N of M. When M is a quasi projective module we obtain that $[N,M]=\{P\in Semp\,(M)\mid N\subseteq P\}$ and $Semp\,(M/N)$ are isomorphic as frames. As an application, we prove that if R is a commutative ring and M a faithful multiplication R-module and $QM\neq M$ for all maximal ideals Q of R, then R has classical Krull dimension if and only if M has classical Krull dimension. Moreover. $cl.K\dim\,(M)=cl.K\dim\,(R)$.

In this paper all rings are associative with an identity, except for some results where R will denote a commutative ring with unity and R-Mod will denote the category of unitary left R-modules. An R-module M is multiplication module if for every submodule N of M, there exists an ideal I of R such that $N=IM$.

Let M and X be R-modules. Then X is said to be M-generated if there exists an R-epimorphism from a direct sum of copies of M onto X. The trace of M in X is defined to be $tr^M(X)=\sum_{f\in Hom_R(M,X)}f(M)$, thus X is M-generated if and only if $tr^M(X)=X$.

If N is a fully invariant submodule of M, we write $N\subseteq_{FI}M$. If N is an essential submodule of M, we write $N\subseteq_{ess}M$. When a module has no non-zero fully invariant proper submodules it is called FI-simple module. An R-module M is a duo module if $N\subseteq_{FI}M$ for all submodules N of M.

Let U be an R-module. If M is an R-module , then U is projective relative to M (U is M-projective) in the case for each epimorphism $g:M\rightarrow N$ and each homomorphism $f:U\rightarrow N$ there is an homomorphism $\widehat{f}:U\rightarrow M$ such that $g\circ\widehat{f}=f$. An R-module M is quasiprojective if M is M-projective.

1 Preliminaries

In this section we provide the necessary material that is needed for the reading of the next sections. We use the product of modules defined in [7] and we show that if M is a multiplication R-module (with R is a ring with commutative multiplication of ideals, in particular when R being a commutative ring), then this product of modules is commutative and associative.

Definition 1.1. [9, Definition 1.1] Let R be a ring and $M \in R\text{-}Mod$. Let K be a submodule of M and $L \in R\text{-}Mod$. We define the product

$$K_M L = \sum \{f(K) \mid f \in Hom(M, L)\}$$

Note that if $M = R$ then the Definition 1.1, then $K_M L$ is the product of left ideals of the ring R.

Note that given a submodule N of M, there exists a submodule $\overline{N} \subset M$ such that $\overline{N}$ is the least fully invariant submodule of M which contains N.

In fact let $\overline{N} = \sum \{f(N) \mid f \in Hom(M, M)\}$, then $\overline{N} = N_M M$. Also notice that if K and L are submodules of M, then

$$\sum \{f(\overline{K}) \mid f \in Hom(M, L)\} = \sum \{f(K) \mid f \in Hom(M, L)\}.$$

Therefore $\overline{K}_M L = K_M L$.

Proposition 1.2. [9, Proposition 1.3] Let $M \in R\text{-}Mod$ and K, K' be submodules of M, then:

1) If $K \subset K'$, then $K_M X \subset K'_M X$ for every $X \in R\text{-}Mod$.

2) If $X \in R\text{-}Mod$ and $Y \subseteq X$, then $K_M Y \subseteq K_M X$.

3) $M_M X = tr^M(X)$ for every $X \in R\text{-}Mod$.

4) $0_M X = 0$ for every $X \in R\text{-}Mod$.

5) $K_M X = 0$ if and only if $f(K) = 0$ for all $f \in Hom(M, X)$.

6) If X, Y are submodules for any module $N \in R\text{-}Mod$, then $K_M X + K_M Y \subseteq K_M(X + Y)$.

7) If $\{K_i\}_{i \in I}$ is a family of submodules of M, then $\left[\sum_{i \in I} K_i \right]_M N = \sum_{i \in I} K_{i\,M} N$.

8) If $\{X_i\}_{i \in I}$ is a family of R-modules, then $K_M \left[\bigoplus_{i \in I} X_i \right] = \bigoplus_{i \in I} K_M X_i$.

Lemma 1.3. Let R be a commutative ring and $M \in R\text{-}Mod$. If M is a multiplication module, then M generates all its submodules.

Proof. Let $N \subseteq M$ be a submodule of M. As M is a multiplication module then there exists an ideal I of R such that $N = IM$. On the other hand we have that $t_r^M(N) = \sum_{f:M \to IM} f(M)$. Since R is a commutative ring, then for each $t \in I$ we can define the morphism $f_t : M \to IM$ such that $f(m) = tm$. Thus $\sum_{f_t} f_t(M) = IM$. But $\sum_{f_t} f_t(M) \subseteq t_r^M(N)$. Thus $N = IM \subseteq t_r^M(N)$. So $t_r^M(N) = N$.

Notice that $t_r^M(N) = \sum_{f:M \to IM} f(M) = M_M N$. So by Lemma 1.3, we have that $M_M N = N$ for all submodules N of M.

Proposition 1.4. Let R be a commutative ring and $M \in R\text{-}Mod$ a multiplication module, then $N_M L = L_M N$ for all submodules N and L of M.

Proof. We have that $N = IM$ and $L = JM$ where I and J are ideals of R. So $N_M L = \sum_{f:M \to L} f(IM) = I \sum_{f:M \to L} f(M) = I \; t_r^M(L) = IL = I(JM) = (IJ)M$. As R is commutative, then $(IJ)M = (JI)M = L_M N$.

Notice that if R is a ring with commutative multiplication of ideals and M is a multiplication module in the sense given by [25] we also obtain the same result of the Proposition 1.4. Also Note that in this case by Proposition 1.2 (7) we have that $N \; _M\sum_{i \in I} K_i = \left[\sum_{i \in I} K_i\right] {}_M N = \sum_{i \in I} (K_{iM} N) = \sum_{i \in I} (N_M K_i)$ for every family of submodules $\{K_i\}_{i \in I}$ of M.

Corollary 1.5. Let R be a commutative ring and M a multiplication R-module. If N, L and K are submodules of M, then $\quad (N_M L)_M K = N_M (L_M K)$.

Proof. It is clear.

Notice that the previous result is not true in general. We consider the example in [10, Remark 1.26] in that example we have that L, K are maximal submodules of $M = E(S)$. Moreover $K_M K = S$ and $S_M K = 0$. Therefore $(K_M K)_M K = S_M K = 0$, but $K_M(K_M K) = K_M S = S$. Hence we have that $(K_M K)_M K \neq K_M(K_M K)$.

Proposition 1.6. Let R be a commutative ring, $M \in R\text{-}Mod$ a faithful multiplication module and P is a prime ideal of R such that $PM \subsetneq M$. If I is an ideal of R such that $IM \subseteq PM$, then $I \subseteq P$.

Proof. As $PM \subsetneq M$, then there exists $x \in M$ and $x \notin PM$. If $a \in I$, then $ax \in IM$. Thus $ax \in PM$, then by [12, Lemma 2.10] we have that $a \in P$ or $x \in PM$. Since $x \notin PM$, then $a \in P$. Therefore $I \subseteq P$.

Corollary 1.7. Let R be a commutative ring, $M \in R\text{-}Mod$ a faithful multiplication module. Suppose that P and P' are prime ideals of R such that $PM \subsetneq M$ and $P'M \subsetneq M$. If $PM = P'M$, then $P = P'$.

Proof. It is clear.

Corollary 1.8. Let R be a commutative ring, $M \in R\text{-}Mod$ a faithful multiplication module such that $PM \subsetneq M$ for all maximal ideal P of R. If Q is a semiprime ideal of R and I is an ideals of R such that $IM \subseteq QM$, then $I \subseteq Q$.

Proof. We know that Q is a semiprime ideal, then $Q = \cap_{\alpha \in \mathcal{L}} P_\alpha$ where every P_α is a prime ideal of R. Hence we obtain that $IM \subseteq QM \subseteq P_\alpha M$ for all $\alpha \in \mathcal{L}$. Now by Proposition 1.6 we have that $I \subseteq P_\alpha$ for all $\alpha \in \mathcal{L}$. So $I \subseteq Q$.

Corollary 1.9. Let R be a commutative ring and $M \in R\text{-}Mod$ a faithful multiplication module such that $QM \subsetneq M$ for all maximal ideals Q of R. Suppose that P and P' are semiprime ideals of R such that $PM = P'M$, then $P = P'$.

Proof. It is clear.

2   Prime and semiprime modules

In this section we use the concepts of prime and semiprime modules defined in [18] and [19] respectively. We give some properties of these modules and we define the radical $\sqrt{N}$ of a submodule N of M. We prove that if M is a faithful multiplication R-module (with R a commutative ring) and $QM \neq M$ for all maximal ideals Q of R, then $\sqrt{IM} = \sqrt{I}M$ for all proper ideals I of R, where $\sqrt{I}$ is the radical of the ideal I. We also prove that if R is a commutative ring and M a faithful multiplication R-module such that $QM \neq M$ for all maximal ideals Q of R, then a proper submodule N of M is semiprime(prime) in M if and only if there exists a semiprime(prime) ideal P of R such that $N = PM$.

We require a goodly number of results from the literature. We include here those results for convenience of the reader.

Definition 2.1 (Raggi-Ríos [19]). Let $M \in R\text{-}Mod$ and $N \neq M$ be a fully invariant submodule of M. We say that N is prime in (or prime submodule of) M if for any K, L fully invariant submodules of M we have that $K_M L \subseteq N$ implies that $K \subseteq N$ or $L \subseteq N$. We say that M is a prime module if 0 is prime in M.

Note that if $M = R$ and I is an ideal of R, then I is prime in R in the sense of Definition 2.1 if and only if I is a prime ideal.

Remark 2.2. In [9, Proposition 1.13] it is shown that if M generates all its fully invariant submodules and N is a maximal fully invariant submodule of M, then N is prime in M. So if R is a commutative ring and M is a multiplication R-module, then by [12, Theorem 2.5] we have that every proper submodule of M is contained in a maximal submodule of M. Moreover if N is a maximal fully invariant submodule of M, then by [9, Proposition 1.13] and Lemma 1.3 we have that N is prime in M.

Notice that if N is a maximal submodule of M, in general N is not prime in M. In order to see this, we consider the example given in [9, Example1.12]. In that example the module $M = E(S)$ is duo but is not a multiplication module. The authors show that M has three maximal submodules but M does not have prime submodules.

Lemma 2.3. Let R be a ring, $M \in R\text{-}Mod$ and $N \subseteq M$ is a fully invariant submodule of M. If N is a submodule of K such that K/N is a fully invariant submodule of M/N, then K is a fully invariant submodule of M.

Proof. It is straightforward.

We require a goodly number of results from the literature. We include here those results for convenience of the reader.

Lemma 2.4 (Raggi-Ríos [19]). Let R be a ring, $M \in R\text{-}Mod$ a quasi projective module and K a fully invariant submodule of M. If N is a submodule of M, then $\dfrac{K + N}{N}$ is a fully invariant submodule of $\dfrac{M}{N}$.

Proposition 2.5 (Raggi-Ríos [19]). Let R be a ring, M a quasi projective R-module and $N \subsetneq M$ is a fully invariant submodule of M. If M/N is a prime module, then N is prime in M.

Proposition 2.6 (Raggi-Ríos [19]). Let R be a ring, M an R-module and $N \subseteq M$ a prime submodule of M, then M/N is a prime module.

Corollary 2.7 (Raggi-Ríos [19]). Let R be a ring, $M \in R$-Mod and N and P submodules of M such that $N \subseteq P$. If P/N is prime in M/N, then M/P is a prime module.

Corollary 2.8 (Raggi-Ríos [19]). Let R be a ring, M an R-module and N a submodule of M. Suppose that P is a proper fully invariant submodule of M such that $N \subseteq P$. If P is prime in M, then P/N is prime in M/N.

Lemma 2.9 (Wisbauer [27]). Let R be a ring, $M \in R$-Mod and N a fully invariant submodule of M. If M is a quasi projective module, then M/N is a quasi projective module.

Corollary 2.10. Let R be a ring, M a quasi projective module, $N \subseteq_{FI} M$ and P a proper submodule of M such that $N \subseteq P$. If $P/N \subseteq_{FI} M/N$ such that M/P is a prime module, then
   $i)$ P/N is prime in M/N.
   $ii)$ P is prime in M.

Proof. $i)$ Apply Lemma 2.9 and Proposition 2.5.
$ii)$ Apply Lemma 2.3 and Proposition 2.5.

Proposition 2.11. Let R be a ring, M a quasi projective module and P a fully invariant submodule of M. The following conditions are equivalents:
   $i)$ P is prime in M.
   $ii)$ For any fully invariant submodules K, L of M containing P and such that $K_M L \subseteq P$, then $K = P$ or $L = P$.
The result in Proposition 2.11 was given in [9, Proposition 1.9]. But we note that it only needs the hypothesis that M is a quasi projective module. The proof is similar.

Definition 2.12 (Raggi-Ríos [20]). Let R be a ring and $M \in R$-Mod. A proper fully submodule N of M is semiprime in M (or a semiprime submodule of M) if for any fully invariant submodule K of M such that $K_M K \subseteq N$, then $K \subseteq N$. We say M is a semiprime module if 0 is semiprime in M.

Notice that if $M = R$, then an ideal I of R is semiprime in the sense of Definition 2.12 if and only if I is a semiprime ideal. Also note that if N is a submodule of M, such that N is an intersection of prime submodules of M, then N is semiprime in M.

Proposition 2.13. Let R be a ring, M a quasi projective multiplication R-module and N a proper submodule of M. Then the following conditions are equivalents:

   $i)$ N is semiprime in M.

ii) If $m \in M$ is such that $Rm_M Rm \subseteq N$, then $m \in N$.

iii) N is an intersection of prime submodules of M.

The result in Proposition 2.13 was given in [12, Proposition 1.11] but with M projective in $\sigma[M]$. As M is a multiplication module, then by [25, Note1.5.] we have that M is a duo module. So we only need the hypothesis that M is a quasi projective module and the proof of Proposition 3.13 is similar to the proof given in [9, Proposition 1.11].

Notice that the condition $ii)$ implies that there exists prime submodules in M.

Remark 2.14. The results obtained in Proposition 2.6, Corollary 2.7 and Corollary 2.8 for prime submodules of M can also be given in terms of semiprime submodules of M.

We know that if R is a commutative ring and I is an ideal of R, then the radical of I, $\sqrt{I}$, is defined as:

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$$

And it can be proven that $\sqrt{I} = \cap \{P \in Spec(R) \mid I \subseteq P\}$.

In the module case we give the following definition:

Definition 2.15. Let R be a ring, M an R-module and N a fully invariant submodule of M. The radical of N in M is

$$\sqrt{N} = \cap \{P \subseteq M \mid P \text{ is a prime in } M \text{ and } N \subseteq P\}$$

If M has no prime submodules P such that $N \subseteq P$, then $\sqrt{N} = M$. In particular $\sqrt{M} = M$.

Remark 2.16. If R is a commutative ring and M is a multiplication module, then by Remark 2.2 we have that every proper submodule N of M is contained in a prime submodule of M. Hence we obtain that $\sqrt{N} \subsetneq M$ for all proper submodules N of M.

Corollary 2.17. Let R be a ring, M a multiplication R-module and N a proper fully invariant submodule of M. If $\sqrt{N} \neq M$, then $\sqrt{N}$ is the minimal semiprime submodule of M such that $N \subseteq \sqrt{N}$.

Proof. As $\sqrt{N} \neq M$, then there exists P a prime module in M such that $N \subseteq P$. So it is clear that $\sqrt{N}$ is a semiprime module. Now let L be a semiprime module in M such that

$N \subseteq L$. By Proposition 2.13 we have that $L = \cap_{i \in I} Q_i$ where Q_i is prime in M for all $i \in I$. Since $N \subseteq L$, then $N \subseteq Q_i$ all $i \in I$. Thus $\sqrt{N} \subseteq L$.

Proposition 2.18. Let R be a ring and M an R-module. Suppose that N and L are fully invariant submodules of M, then the following conditions hold:

i) If $N \subseteq L$, then $\sqrt{N} \subseteq \sqrt{L}$.

ii) $\sqrt{N} = \sqrt{\sqrt{N}}$.

iii) $\sqrt{N + L} = \sqrt{\sqrt{N} + \sqrt{L}}$.

iv) $\sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L}$.

v) $\sqrt{N_M L} \subseteq \sqrt{N} \cap \sqrt{L}$.

Proof. They are straightforward.

If we define $N^2 = N_M N$. Then by induction, for any integer $n > 2$, we define $N^n = N_M N^{n-1}$. Note that if N is prime in M, then $\sqrt{N^n} = N$.

Definition 2.19. If R is a commutative ring and M is a multiplication R-module, an element $m \in M$ is M-nilpotent if $(Rm)^n = 0$ for some $n > 0$. The M-nilradical $\mathcal{N}(M)$ of M is the set of all M-nilpotent elements in M.

Proposition 2.20. If R is a commutative ring and M is a multiplication R-module, then the M-nilradical $\mathcal{N}(M)$ is a submodule of M.

Proof. If $m \in \mathcal{N}(M)$ and $r \in R$, then $R(rm) = (Rr)m \subseteq Rm$. As $(Rm)^n = 0$ for some $n > 0$, then $(R(rm))^n \subseteq (Rm)^n = 0$. Hence $rm \in \mathcal{N}(M)$. Now let $m_1, m_2 \in \mathcal{N}(M)$, then $(Rm_1)^{n_1} = 0$ and $(Rm_2)^{n_2} = 0$ for some $n_1 > 0$ and $n_2 > 0$. As M is a multiplication module, then by Proposition 1.4. we have that $(Rm_2)_M (Rm_1) = (Rm_1)_M (Rm_2)$. Thus we can use the binomial theorem. So $(Rm_1 + Rm_2)^{n_1 + n_2 - 1}$ is a sum of integer multiples of products $(Rm_1)^r$ $(Rm_2)^s$, where $r + s = m + n - 1$. We cannot have both $r < n_1$ and $s < n_2$. Hence each of these products vanishes and therefore $(Rm_1 + Rm_2)^{n_1 + n_2 - 1} = 0$. Thus $m_1 + m_2 \in \mathcal{N}(M)$. So $\mathcal{N}(M)$ is a submodule of M.

Proposition 2.21. Let R be a commutative ring and M a non zero multiplication R-module, then the following conditions hold:

i) $\mathcal{N}(M) \subseteq \sqrt{0}$.

ii) If N is a prime submodule of M, then $ann(M/N)$ is a prime ideal of R.

iii) If M is a faithful multiplication module and Q is an ideal prime of R such that $QM \neq M$, then QM is prime in M.

Proof. *i*) As $M \neq 0$, then $\sqrt{0} \neq M$. So $\sqrt{0}$ is the intersection of all prime submodules of M. If $m \in \mathcal{N}(M)$, then there exists $n > 0$ such that $(Rm)^n = 0$. So $(Rm)^n \subseteq P$ for all prime submodule P of M. Thus $Rm \subseteq P$ for all prime submodules P of M. Hence $m \in \sqrt{0}$. Thus $\mathcal{N}(M) \subseteq \sqrt{0}$.

ii) Suppose that I and J are ideals of R such that $IJ \in ann(M/N)$. So $(IJ)M \subseteq N$. Now we consider the modules $K = IM$ and $L = JM$. So by proof of Proposition 1.4 we have that $K_M L = (IJ)M \subseteq N$. As N is prime in M, then $IM = K \subseteq N$ or $JM = L \subseteq N$. Hence $I(M/N) = 0$ or $J(M/N) = 0$. Thus $I \subseteq ann(M/N)$ or $J \subseteq ann(M/N)$.

iii) Let K and L be a submodules of M such that $K_M L \subseteq QM$. Since M is a multiplication module, then there exists I and J ideals of R such that $K = IM$ and $L = JM$. Hence $(IJ)M = (IM)_M(JM) \subseteq QM$. So by Proposition 1.6 we have that $IJ \subseteq Q$. As Q is a prime ideal, then $I \subseteq Q$ or $J \subseteq Q$. Thus $IM \subseteq QM$ or $JM \subseteq QM$. So $K \subseteq QM$ or $L \subseteq QM$. Thus QM is prime in M.

Notice that if $N = QM$ where Q is a prime ideal of R, then $Q = ann(M/N)$. In fact as $QM = N = ann(M/N)M$, then by Corollary 1.7 we have that $Q = ann(M/N)$. Also note that if M is a finitely generated module, then by [12, Theorem 3.1] we have that $QM \neq M$ for all proper ideals Q of R. So if M is as in *iii*) and M is finitely generated, then QM, is a submodule prime in M for all prime ideals Q of R.

Corollary 2.22. Let R be a commutative ring, $M \in R\text{-}Mod$ is a faithful multiplication module and Q is an ideal prime of R. Suppose that $QM \neq M$, then $Q = ann(M/QM)$.

Proof. By Proposition 2.21 we have that $N = QM$ is a prime submodule of M and $ann(M/N)$ is a prime ideal of R. As $QM = N = ann(M/N)M$, then by Corollary 1.7 we have that $Q = ann(M/N) = ann(M/QM)$.

Note that we can obtain similar results to the Proposition 2.21 $ii)$, $iii)$ and Corollary 2.22 in terms of semiprime modules in the following propositions.

Proposition 2.23. Let R be a commutative ring and M a multiplication R-module, then the following conditions hold:

$i)$ If N is a semiprime submodule of M, then $ann\,(M/N)$ is a semiprime ideal of R.

$ii)$ If M is a faithful multiplication R-module and Q is a semiprime ideal of R such that $QM \neq M$, then QM is semiprime in M.

Corollary 2.24. Let R be a commutative ring, $M \in R\text{-}Mod$ a faithful multiplication module and Q a semiprime ideal of R. Suppose that $QM \neq M$, then $Q = ann\,(M/QM)\,M$.

Proposition 2.25. Let R be a commutative ring and M a faithful multiplication R-module. Suppose that $QM \neq M$ for all maximal ideals Q of R. If $P = \cap_{\alpha \in \mathcal{L}} P_\alpha$ with P_α prime ideal of R for every $\alpha \in \mathcal{L}$, then $PM = \cap_{\alpha \in \mathcal{L}} (P_\alpha M)$.

Proof. As M is a multiplication module, then by Remark 2.16 we have that $P_\alpha M \neq M$ for all $\alpha \in \mathcal{L}$. If we put $N = PM$ and $N' = \cap_{\alpha \in \mathcal{L}} (P_\alpha M)$, then $PM \subseteq P_\alpha M$ for all $\alpha \in \mathcal{L}$. Thus $N \subseteq N'$. On the other hand, by Proposition 2.13 we have that N' is semiprime in M. Moreover we know that $N' = ann\,(M/N')\,M$. Thus $(\cap_{\alpha \in \mathcal{L}} P_\alpha)\,M \subseteq ann\,(M/N')\,M$. Now by Proposition 2.23 we have that $ann\,(M/N')$ is a semiprime ideal of R. Thus by Proposition 1.8 we have that $\cap_{\alpha \in \mathcal{L}} P_\alpha \subseteq ann\,(M/N')$. As $ann\,(M/N')\left(\frac{M}{N'}\right) = 0$, then $ann\,(M/N')\,M \subseteq N' = \cap_{\alpha \in \mathcal{L}} (P_\alpha M)$. Hence $ann\,(M/N')\,M \subseteq P_\alpha M$ for all $\alpha \in \mathcal{L}$. So by Proposition 1.6 we have that $ann\,(M/N') \subseteq P_\alpha$ for all $\alpha \in \mathcal{L}$. Thus $ann\,(M/N') \subseteq \cap_{\alpha \in \mathcal{L}} P_\alpha$. So we have that $\cap_{\alpha \in \mathcal{L}} P_\alpha = ann\,(M/N')$. Thus $PM = (\cap_{\alpha \in \mathcal{L}} P_\alpha)\,M = ann\,(M/N')\,M$.

Corollary 2.26. Let R be a commutative ring and M a faithful multiplication R-module. Suppose that $QM \neq M$ for all maximal ideals Q of R, then a proper submodule N of M is semiprime (prime) in M if and only if there exists a semiprime (prime) ideal P of R such that $N = PM$.

Proof. $\Rightarrow$] Suppose that N is semiprime(prime) in M, then by Proposition 2.13, we have that $N = \cap_{\alpha \in \mathcal{L}} N_\alpha$ where every N_α is prime in M. Now by Proposition 2.23 (Proposition 2.21) we know that $ann\,(M/N_\alpha)$ semiprime(prime) ideal of R and $N_\alpha = ann\,(M/N_\alpha)\,M$. Thus $N = \cap_{\alpha \in \mathcal{L}} N_\alpha = N = \cap_{\alpha \in \mathcal{L}} [ann\,(M/N_\alpha)\,M]($ or $N = ann\,(M/N)\,M)$. By Proposition 2.25 we have that $N = [\cap_{\alpha \in \mathcal{L}} ann\,(M/N_\alpha)]\,M$ (or $N = ann\,(M/N)\,M$). Moreover $\cap_{\alpha \in \mathcal{L}} ann\,(M/N_\alpha)$ $(ann\,(M/N))$ is a semiprime (prime) ideal of R.

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$\Leftarrow$] As P is a semiprime(prime) ideal of R, then $P = \cap_{i \in \mathcal{T}} P_i$ with every P_i is a prime ideal of R. Then by Proposition 2.25 we have that $N = PM = \cap_{i \in \mathcal{T}} (P_i M)$. By Proposition 2.21 we have that $P_i M$ is prime in M. Therefore N is semiprime in M.

Theorem 2.27. Let R be a commutative ring and M a faithful multiplication R-module and $QM \neq M$ for all maximal ideal Q of R. Then $\sqrt{IM} = \sqrt{I}M$ for all proper ideals I of R. Where $\sqrt{I}$ is the prime radical of I.

Proof. Suppose that $\sqrt{I} = \cap_{\alpha \in \mathcal{L}} P_\alpha$ where every P_α is a prime ideal of R with $I \subseteq P_\alpha$ for all $\alpha \in \mathcal{L}$. Now by Proposition 2.25 we have that $\sqrt{I}M = \cap_{\alpha \in \mathcal{L}} (P_\alpha M)$. As $P_\alpha M$ is prime in M, then $\sqrt{I}M$ is semiprime in M. Since $I \subseteq \sqrt{I}$, then $IM \subseteq \sqrt{I}M$. Therefore $\sqrt{IM} \subseteq \sqrt{I}M$. On the other hand if we put $\mathcal{N} = \{N' \subset M \mid N' \text{ is prime in } M \text{ and } IM \subseteq N'\}$, then $\sqrt{IM} = \cap_{N' \in \mathcal{N}} N'$. Let $N' \in \mathcal{N}$. So $N' = P'M$ with P' prime ideal of R. Thus $IM \subseteq P'M$. By Proposition 1.6 we have that $I \subseteq P'$. Therefore $\sqrt{I} \subseteq P'$. Hence $\sqrt{I}M \subseteq P'M = N'$ for all $N' \in \mathcal{N}$. Thus $\sqrt{I}M \subseteq \sqrt{IM}$.

Notice that if N is a proper submodule of M, then $N = IM$ for some proper ideal I of R. So by Theorem 2.27 we have that $\sqrt{N} = \sqrt{I}M$.

3   Zariski Topology for Multiplication Modules

In this section we give the Zariski Topology for a module multiplication M. We describe open sets and closed sets of this topology and we give a basis of open sets for the Zariski topology.

We denote $Spec(M) = \{P \mid P \text{ is a prime submodule of } M\}$.

Several of the followings results have been given recently. We include here those results for convenience of the reader.

Proposition 3.1 . (Jawad [13], Jawad-Lomp [14]). Let R be a ring and M a multiplication R-module, then $(Spec(M), \mathcal{T})$ is a topological space,

where $\mathcal{T} = \{\mathcal{U}(N) \mid N \in Sub(M)\}$ is the topology and $\mathcal{U}(N) = \{P \in Spec(M) \mid N \nsubseteq P\}$ are open sets.

Remark 3.2. As $\mathcal{U}(N) = \mathcal{U}\left(\sqrt{N}\right)$ for all $N \in Sub(M)$, then:

$\mathcal{T} = \{\mathcal{U}(N) \mid N \in Semp(M) \cup \{M\}\}$. Thus we can consider the open sets as $\mathcal{U}(N)$ with N semiprime in M or $N = M$.

Following to [4] we say that $\mathcal{T}$ is the Zariski topology and $Spec\,(M)$ is called the prime spectrum of M. The topological space $(Spec\,(M)\,,\,\mathcal{T})$ will be denoted by $\mathbf{Z}\,(M)$.

Lemma 3.3 (Jawad [13]. Jawad-Lomp [14]). Let R be a ring and M an R-module. If N and L are submodules of M, then the following conditions hold:

i) $\mathcal{U}\,(L) \cap \mathcal{U}\,(N) = \mathcal{U}\,(L_M N)$.

ii) $\mathcal{U}\,(L) = \varnothing$ if and only if $L \subseteq \sqrt{0}$.

iii) $\mathcal{U}\,(L) = Spec\,(M)$ if and only if $L = M$.

iv) $\mathcal{U}\,(L) = \mathcal{U}\,(N)$ if and only if $\sqrt{L} = \sqrt{N}$.

Notice that if M is a semiprime module, then $0 = \cap_{p \in Spec(M)} P$. Thus $\sqrt{0} = 0$.

Let R be a commutative ring and M a multiplication R-module. For each subset E of M, we denote $\mathcal{V}\,(E) = \{P \in Spec\,(M) \mid E \subseteq P\}$. Notice that $\{N_i\}_{i \in I}$ is a family of submodules of M. Moreover we have that $\mathcal{V}\,(\cup_{i \in I} N_i) = \mathcal{V}\left(\sum_{i \in I} N_i\right)$.

Proposition 3.4. (Jawad [13]. Jawad-Lomp [14]). Let R be a ring and M an R-module, then the following conditions hold:

i) If E is a subset of M and $\langle E \rangle = \sum_{m \in E} Rm$, then $\mathcal{V}\,(E) = \mathcal{V}\,(\langle E \rangle) = \mathcal{V}\left(\sqrt{\langle E \rangle}\right)$.

ii) $\mathcal{V}\,(0) = Spec\,(M)$ and $\mathcal{V}\,(M) = \varnothing$.

iii) If $\{E_i\}_{i \in I}$ is a family of subsets of M, then $\mathcal{V}\,(\cup_{i \in I} E_i) = \cap_{i \in I} \mathcal{V}\,(E_i)$.

iv) If N and L are submodules of M then $\mathcal{V}\,(N \cap L) = \mathcal{V}\,(N_M L) = \mathcal{V}\,(N) \cup \mathcal{V}\,(L)$.

Notice that if $\langle E \rangle \in Sub\,(M)$, then the complement $\mathcal{V}\,(\langle E \rangle)^C$ of $\mathcal{V}\,(\langle E \rangle)$ is the set $\mathcal{U}\,(\langle E \rangle)$. Thus the results of Proposition 3.4 show that the sets $\mathcal{V}\,(\langle E \rangle)$ satisfy axioms for closed sets in the Zariski topology. We also note that $\mathcal{V}\,(\langle \{m\} \rangle)^C = \mathcal{V}\,(Rm)^C = \mathcal{U}\,(Rm)$ for all $m \in M$.

Proposition 3.5. (Jawad [13]). Let R be a ring and M an R-module. Then $\mathcal{B} = \{\mathcal{U}\,(Rm) \mid m \in M\}$ is a basis of open sets for the Zariski topology.

4   Compact, Irreducible and Dense subspaces

In this section we characterize compact sets of the form $\mathcal{U}\,(N)$ in terms of finitely generated submodules of M. We also characterize irreducible sets of the form $\mathcal{U}\,(N)$ in terms of finitely uniform submodules of M.

Proposition 4.1.(Jawad [13]. Jawad-Lomp [14]) Let R be a commutative ring and M a multiplication R-module. Then the following conditions are equivalents:

i) M is finitely generated.

ii) The topological space $\mathbf{Z}(M)$ is compact (that is, every open covering of $Spec(M)$ has a finite subcover).

Corollary 4.2. Let R be a commutative ring and M a multiplication R-module. If M is finitely generated and N is a submodule of M such that N is a direct summand of M, then $\mathcal{U}(N)$ is compact in $\mathbf{Z}(M)$.

Proof. As N is a direct summand of M then there exists a submodule L of M such that $N \oplus L = M$. Now let $\{\mathcal{U}(Rm_i)\}_{i \in I}$ be an open cover of $\mathcal{U}(N)$ and $\{\mathcal{U}(Rm_j)\}_{i \in J}$ an open cover of $\mathcal{U}(L)$. We can suppose that $\mathcal{U}(N) \cap \mathcal{U}(Rm_i) \neq \varnothing$ and $\mathcal{U}(L) \cap \mathcal{U}(Rm_j) \neq \varnothing$ for all i, j such that $i \in I$ and $j \in J$. So it is clear that $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i \in I}$ and $\{\mathcal{U}(L) \cap \mathcal{U}(Rm_j)\}_{i \in J}$ are open covers of $\mathcal{U}(N)$ and $\mathcal{U}(L)$ respectively. We claim that $[\mathcal{U}(N) \cap \mathcal{U}(Rm_i)] \cap [\mathcal{U}(L) \cap \mathcal{U}(Rm_j)] = \varnothing$ for all i, j such that $i \in I$ and $j \in J$. In fact let $P \in [\mathcal{U}(N) \cap \mathcal{U}(Rm_i)] \cap [\mathcal{U}(L) \cap \mathcal{U}(Rm_j)]$, then $P \in \mathcal{U}(N) \cap \mathcal{U}(L)$. So $N \nsubseteq P$ and $L \nsubseteq P$. As P is a prime submodule of M then $N_M L \nsubseteq P$. Since M is a duo module, then $N_M L \subseteq N \cap L = 0$. Therefore $N_M L \subseteq P$ for all, $P \in Spec(M)$ is a contradiction. Hence $[\mathcal{U}(N) \cap \mathcal{U}(Rm_i)] \cap [\mathcal{U}(L) \cap \mathcal{U}(Rm_j)] = \varnothing$.

Now $Spec(M) = \mathcal{U}(M) = \mathcal{U}(N \oplus L) = \mathcal{U}(N) \cup \mathcal{U}(L)$, then $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i \in I} \cup \{\mathcal{U}(L) \cap \mathcal{U}(Rm_j)\}_{i \in J}$ is an open cover of $Spec(M)$. By Proposition 4.1 we have that $Spec(M)$ is compact. So the cover $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i \in I} \cup \{\mathcal{U}(L) \cap \mathcal{U}(Rm_j)\}_{j \in J}$ has a finite subcover. Let $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i=1}^{n} \cup \{\mathcal{U}(L) \cap \mathcal{U}(Rm_j)\}_{j=1}^{r}$ be a finite subcover such that

$\cup_{i=1}^{n} (\mathcal{U}(N) \cap \mathcal{U}(Rm_i)) \cup \{\mathcal{U}(L) \cap \mathcal{U}(Rm_j)\}_{i=1}^{r}$.

As $[\mathcal{U}(N) \cap \mathcal{U}(Rm_i)] \cap [\mathcal{U}(L) \cap \mathcal{U}(Rm_j)] = \varnothing$ for all $i = 1, 2, .., n$ and $j = 1, 2, ..., r$, then $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i=1}^{n}$ is a finite subcover of $\{\mathcal{U}(N) \cap \mathcal{U}(Rm_i)\}_{i \in I}$.

So $\mathcal{U}(N) = \cup_{i=1}^{n} [\mathcal{U}(N) \cap \mathcal{U}(Rm_i)] \subseteq \cup_{i=1}^{n} \mathcal{U}(Rm_i)$. Thus $\mathcal{U}(N)$ is compact.

Proposition 4.3. (Jawad [13]. Jawad-Lomp [14]). Let R be a commutative ring and M a multiplication R-module. If N is a submodule of M such that $\mathcal{U}(N)$ is compact, then there exists a finitely generated submodule L of N such that $\mathcal{U}(N) = \mathcal{U}(L)$.

Proposition 4.4. (Jawad [13]. Jawad-Lomp [14]). Let R be a commutative ring and M a multiplication R-module. If E is an open subset of $Spec(M)$ such that E is compact, then there exists a finitely generated submodule L of M such that $E = \mathcal{U}(L)$.

The following definition was given in [8, Commutative Algebra II-4-1].

Definition 4.5. A topological space X is said to be irreducible if $X \neq \varnothing$ and that the intersection of two non-empty open sets of X be always non-empty. A non-empty subset Y of X is an irreducible set in X if the subspace Y of X is irreducible.

Proposition 4.6. Let R be a commutative ring and M a multiplication R-module. Suppose that M is a semiprime module. If N is a submodule of M such that $\mathcal{U}(N)$ is an irreducible set in $\mathbf{Z}(M)$, then N is a uniform module.

Proof. Let $K \neq 0$ and $L \neq 0$ be proper submodules of N. We claim that $\mathcal{U}(K) \neq \varnothing$ and $\mathcal{U}(L) \neq \varnothing$. In fact if $\mathcal{U}(K) = \varnothing$, then by Lemma 3.3 $ii)$ we have that $K \subseteq \sqrt{0} = \cap_{P \in Spec(M)} P$. As M is a semiprime module, then 0 is a semiprime submodule of M. So $\sqrt{0} = \cap_{P \in Spec(M)} P = 0$. Thus $K = 0$ it is a contradiction. Analogously $\mathcal{U}(L) \neq \varnothing$. Since $\mathcal{U}(K) \subseteq \mathcal{U}(N)$; $\mathcal{U}(L) \subseteq \mathcal{U}(N)$ and $\mathcal{U}(N)$ is irreducible, then $\mathcal{U}(K) \cap \mathcal{U}(L) \neq \varnothing$. By Lemma 3.3 $i)$ we have that $\mathcal{U}(K_M L) \neq \varnothing$. Hence $K_M L \neq 0$. Since M is a duo module, then $N_M L \subseteq N \cap L$. Thus $N \cap L \neq 0$. So N is a uniform module.

Notice that if R is a commutative ring and M is a multiplication R-module such that $\mathcal{U}(M) = Spec(M)$ is an irreducible set in $\mathbf{Z}(M)$, then M is a uniform module.

Proposition 4.7. Let R be a commutative ring and M a multiplication R-module. If N is a uniform submodule of M and $\sqrt{0}$ is prime in M, then $\mathcal{U}(N)$ is an irreducible set in $\mathbf{Z}(M)$.

Proof. We denote $\sqrt{0} = Q$. Let $\mathcal{U}(K) \neq \varnothing$ and $\mathcal{U}(L) \neq \varnothing$ be open sets such that $\mathcal{U}(K) \subseteq \mathcal{U}(N)$ and $\mathcal{U}(L) \subseteq \mathcal{U}(N)$. So $N \neq 0$ and $L \neq 0$. By Lemma 3.3 $i)$ we have that $\mathcal{U}(L) \cap \mathcal{U}(L) = \mathcal{U}(N_M L)$. Now if $\mathcal{U}(K_M L) = \varnothing$, then $K_M L \subseteq Q$. As Q is prime in M, then $K \subseteq Q$ or $L \subseteq Q$. On the other hand we have that $Q = \sqrt{0} = \cap_{P \in Spec(M)} P$. Hence $Q \subseteq P$ for all $P \in Spec(M)$. Thus $\mathcal{U}(Q) = \varnothing$. Since $K \subseteq Q$ or $L \subseteq Q$, then $\mathcal{U}(K) \subseteq \mathcal{U}(Q) = \varnothing$ or $\mathcal{U}(L) \subseteq \mathcal{U}(Q) = \varnothing$, is a contradiction. Thus $\mathcal{U}(L) \cap \mathcal{U}(L) = \mathcal{U}(N_M L) \neq \varnothing$. Hence $\mathcal{U}(N)$ is irreducible.

Remark 4.8. If M is a prime and duo module, then M is a uniform module. In fact let N and L be submodules of M such that $N \cap L = 0$. As M is a duo module, then $N_M L \subseteq N \cap L = 0$. Since M is a prime module, then $N = 0$ or $L = 0$. So M is a uniform module. Moreover, when M is a prime multiplication R-module we know that 0 is prime in M. So $\sqrt{0} = 0$. Thus $\mathcal{U}(N)$ is an irreducible set in $\mathbf{Z}(M)$ for all non-zero submodules N of M.

Definition 4.9. A subset B of a topological space X is said to be dense in X if $U \cap B \neq \varnothing$ for all open sets $\varnothing \neq U$ of X.

Proposition 4.10. Let R be a commutative ring and M a multiplication R-module. Suppose that $\sqrt{0}$ is a prime submodule of M. If $N \neq 0$ is a uniform submodule of M such that $\mathcal{U}(N) \neq \varnothing$, then $\mathcal{U}(N)$ is dense in the topological space $\mathbf{Z}(M)$.

Proof. Let $\mathcal{U}(L) \neq \varnothing$ be an open set of the topological space $\mathbf{Z}(M)$. Suppose that $\mathcal{U}(N) \cap \mathcal{U}(L) = \varnothing$. Then by Lemma 3.3 $i)$ we have that $\mathcal{U}(N_M L) = \varnothing$. Hence $N_M L \subseteq \sqrt{0}$. As $\sqrt{0}$ is a prime submodule, then $N \subseteq \sqrt{0}$ or $L \subseteq \sqrt{0}$. Hence $\mathcal{U}(N) = \varnothing$ or $\mathcal{U}(L) = \varnothing$ is a contradiction. Therefore $\mathcal{U}(N) \cap \mathcal{U}(L) \neq \varnothing$. So $\mathcal{U}(N)$ is dense.

Note that Proposition 4.10 is not true in general. We consider the following example:

Example 4.11 . Let $R = \mathbb{Z}$, p be a prime number and $M = \mathbb{Z}_{p^n}$ with $n \geq 2$. We know that M is a $\mathbb{Z}$-multiplication module. It is clear that $p\mathbb{Z}_{p^n}$ is a uniform submodule of M. Moreover $p\mathbb{Z}_{p^n}$ is the only one prime submodule of M. So $\sqrt{p\mathbb{Z}_{p^n}} = p\mathbb{Z}_{p^n}$. Thus we have that $\mathcal{U}(p\mathbb{Z}_{p^n}) = \varnothing$. Therefore $\mathcal{U}(p\mathbb{Z}_{p^n})$ is not dense in the topological space $\mathbf{Z}(M)$.

Corollary 4.12. Let R be a commutative ring and M a prime multiplication R-module. If $N \neq 0$ is a uniform submodule of M, then $\mathcal{U}(N)$ is dense in the topological space $\mathbf{Z}(M)$.

Proof. We claim that $\mathcal{U}(N) \neq \varnothing$. In fact if $\mathcal{U}(N) = \varnothing$, then by Lemma 3.3 $ii)$ we have that $N \subseteq \sqrt{0}$. Now as M is a prime module, then 0 is a prime submodule of M. Thus $\sqrt{0} = \cap_{P \in Spec(M)} P = 0$. Hence $N = 0$ is a contradiction. Thus $\mathcal{U}(N) \neq \varnothing$. So by Proposition 4.10 we have the result.

Note that in Example 4.11, the module $M = \mathbb{Z}_{p^n}$ is not a prime module.

5 Main Results and Applications

In this section we prove that $\{Semp(M), \wedge, \vee\}$ is a frame for every ring R and every multiplication R-module M. We also prove that if R is a commutative ring and M a multiplication R-module, then $Sub(M)$ is a bilateral quantal. Moreover we prove $Semp(M/N)$ is a spatial frame for all submodules N of M. When M is a quasi projective module we obtain that $[N, M] = \{P \in Semp(M) \mid N \subseteq P\}$ and $Semp(M/N)$ are isomorphic as frames. On the other hand when M is a faithful multiplication R-module and $QM \neq M$ for all maximal

ideals Q of R, then $i)$ The topological spaces $Spec\,(R)$ and $Spec\,(M)$ are homeomorphic $ii)$ $Semp\,(R) \cong Semp\,(M)$ as frames $iii)$ $cl.K\dim\,(M) = cl.K\dim\,(R)$. As a special result we obtain that $\Psi\,(M) = \{N \subseteq M \mid N + Ann_M\,(Rn) = M,\, \forall n \in N\}$ is a spatial frame for every multiplication R-module M. The set $\Psi\,(M)$ was studied by [17]. Finally we show that if R is a ring and $\mathbf{Z}\,(M)$ is a noetherian topological space, then M has a classical Krull dimension.

Remark 5.1. When R is a commutative ring and M is a multiplication R-module, then by Proposition 1.2 (7) we have that $N\,_M\sum_{i \in I} K_i = \left[\sum_{i \in I} K_i\right]_M N = \sum_{i \in I} (K_{iM} N) = \sum_{i \in I} (N_M K_i)$ for every $\{K_i\}_{i \in I}$ family of submodules of M and for all submodules N of M.

For the definition of frame and quantal see [17].

Proposition 5.2. Let R be a commutative ring and M a multiplication R-module, then $\{Sub\,(M),\ \leq,\ \vee,\ \wedge,\ M,\ 0,\ _{-M-}\}$ is a bilateral quantal.

Proof. It is clear that $\{Sub\,(M),\ \leq,\ \vee,\ \wedge,\ M,\ 0\}$ is a complete lattice where "$\leq$" denotes $\subseteq$. As R is a commutative ring, then by Corollary 1.5 we have that the product $_{-M-} : Sub(M) \times Sub(M) \rightarrow Sub(M)$ is associative. Moreover by Remark 5.1 we obtain

that $N_M\sum_{i \in I} K_i = \sum_{i \in I} (N_M K_i)$ for every $\{K_i\}_{i \in I}$ family of submodules of M and for all submodules N and L of M. Moreover by Proposition 1.2 (7) we have that:

$\left[\sum_{i \in I} K_i\right]_M N = \sum_{i \in I} (K_{iM} N)$. Thus by [17, Definition 2.4] we have that

$\{Sub\,(M),\ \leq,\ \vee,\ \wedge,\ M,\ 0,\ _{-M-}\}$ is a quantal. Now by Lemma 1.3 we have that $M_M N = t_r^M\,(N) = N$. As M is a duo module, then N is a fully invariant submodule of M. So $N_M M = N$. Hence $\{Sub\,(M),\ \leq\ \vee,\ 0,\ _{-M-}\}$ is a bilateral quantal.

We denote $Semp\,(M) = \{N \subset M \mid N$ is semiprime in $M\} \cup \{M\}$. It is easy to prove that $N \wedge N' = N \cap N'$ and $N \vee N' = \sqrt{N + N'}$ are the meet and join of lattice $Semp\,(M)$. Moreover this lattice is complete.

Theorem 5.3. Let R be a ring and M a multiplication R-module. Then $\{Semp\,(M),\ \wedge,\ \vee\}$ is a frame.

Proof. We know that $\{Semp\,(M),\ \wedge,\ \vee\}$ is a complete lattice. Now let $N \in Semp\,(M)$ and $\{N_i\}_{i \in \mathcal{I}}$ be a family of submodules in $Semp\,(M)$. We will prove that $N \wedge (\vee_{i \in \mathcal{I}} N_i) = \vee_{i \in \mathcal{I}} (N \wedge N_i)$.

As $N \wedge (\vee_{i \in \mathcal{I}} N_i) = N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i}\right)$ and $\vee_{i \in \mathcal{I}} (N \wedge N_i) = \sqrt{\sum_{i \in \mathcal{I}} (N \cap N_i)}$. If $N = M$, then we have the result. Suppose that $N \subsetneq M$. It is clear that $N \cap N_j \subseteq N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i}\right)$

for all $j \in \mathcal{I}$. Thus $\sum_{i \in \mathcal{I}} (N \cap N_i) \subseteq N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right)$. By Proposition 2.13 we have that $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right)$ is an intersection of prime submodules of M. So $\sqrt{\sum_{i \in \mathcal{I}} (N \cap N_i)} \subseteq N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right)$. Now let P prime in M such that $\sum_{i \in \mathcal{I}} (N \cap N_i) \subseteq P$. Thus $N \cap N_i \subseteq P$ for all $i \in \mathcal{I}$. Since N is a fully invariant submodule of M, we have that $N_M N_i \subseteq N \cap N_i$. So $N_M N_i \subseteq P$. As P is prime in M, then $N \subseteq P$ or $N_i \subseteq P$. If $N \subseteq P$, then $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right) \subseteq P$. Hence $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right) \subseteq \sqrt{\sum_{i \in \mathcal{I}} (N \cap N_i)}$. If $N \nsubseteq P$, then $N_i \subseteq P$ for all $i \in \mathcal{I}$. Thus $\sum_{i \in \mathcal{I}} N_i \subseteq P$. So $\sqrt{\sum_{i \in \mathcal{I}} N_i} \subseteq P$. Therefore $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right) \subseteq P$. Hence $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right) \subseteq \sqrt{\sum_{i \in \mathcal{I}} (N \cap N_i)}$. Therefore $N \cap \left(\sqrt{\sum_{i \in \mathcal{I}} N_i} \right) = \sqrt{\sum_{i \in \mathcal{I}} (N \cap N_i)}$. So $N \wedge (\vee_{i \in \mathcal{I}} N_i) = \vee_{i \in \mathcal{I}} (N \wedge N_i)$.

Note that if M is a multiplication R-module and $N \in Semp\,(M)$, then the set $[N, M] = \{P \in Semp\,(M) \mid N \subseteq P\}$ is a subframe of $Semp\,(M)$.

Remark 5.4. If R is a ring we know that every semiprime ideal is an intersection of prime ideals of R. Therefore we have that $\{Semp\,(R), \, \wedge, \, \vee\}$ is a frame for every ring R.

We denote $\mathcal{O}\,(Spec\,(M)) = \{\mathcal{T}, \, \subseteq, \cup, \cap\}$ the frame of open subsets of $Spec\,(M)$, where $\mathcal{T}$ is the Zariski topology of $Spec\,(M)$.

Theorem 5.5. Let R be a ring and M a multiplication R module, then $Semp\,(M) \cong \mathcal{O}\,(Spec\,(M))$ as frames.

Proof. We define $\Psi : Semp\,(M) \rightarrow \mathcal{O}\,(Spec\,(M))$ such that $\Psi\,(N) = \mathcal{U}\,(N)$. We claim that Ψ is an order isomorphism. In fact suppose that $N_1 \subseteq N_2$. If $P \in \mathcal{U}\,(N_1)$, then $N_1 \nsubseteq P$. Thus $N_2 \nsubseteq P$. So $\Psi\,(N_1) \subseteq \Psi\,(N_2)$. Moreover if $\Psi\,(N_1) = \Psi\,(N_2)$, then $N_1 = N_2$. Thus Ψ is injective. Now let $\mathcal{U}\,(N) \in \mathcal{O}\,(Spec\,(M))$, by Remark 3.2 we have that N is a semiprime in M. So Ψ is surjective. Therefore Ψ is bijective and $\Psi^{-1}\,(\mathcal{U}\,(N)) = N$. Now suppose that $\mathcal{U}\,(N_1) \subseteq \mathcal{U}\,(N_2)$. Thus if P is prime in M such that $N_2 \subseteq P$, then $N_1 \subseteq P$. As N_1 and N_2 are semiprime modules, then by Proposition 2.13 we have that $N_1 \subseteq N_2$. So Ψ is an order isomorphism, now by [23, Chapter III Proposition 1.1] we have that Ψ is a lattice isomorphism. Hence Ψ is a frame isomorphism.

Definition 5.6. Let $\mathcal{F}$ be a frame. It is said that $\mathcal{F}$ is spatial, if $\mathcal{F}$ is isomorphic to $\mathcal{O}\,(X)$ the frame of open sets of some topological space X.

Corollary 5.7. Let R be a ring and M a multiplication R-module, then $Semp\,(M/N)$ is a spatial frame for all submodules N of M.

Proof. As M is a multiplication R-module, then M/N is a multiplication module. So by Theorem 5.5 we have that $Semp\,(M/N) \cong \mathcal{O}\,(Spec\,(M/N))$ for all submodules N of M. Hence $Semp\,(M/N)$ is a spatial frame for all submodules N of M.

Lemma 5.8. Let R be a ring and M a multiplication R-module. Suppose that M is quasi projective and N is a submodule of M, then $Semp\,(M/N) = \{P/N \mid P \in Semp\,(M)\}$.

Proof. As M is a multiplication module, then M is a duo module. By Remark 2.14 we have that $Semp\,(M/N) = \{P/N \mid P \in Semp\,(M)\}$.

Proposition 5.9. Let R be a ring and M a multiplication R-module. Suppose that M is quasi projective and $N \in Semp\,(M)$, then $[N, M]$ and $Semp\,(M/N)$ are isomorphic as frames.

Proof. By Lemma 5.8 we have that $Semp\,(M/N) = \{P/N \mid P \in Semp\,(M)\}$. So we can define the morphism $\phi:[N, M] \to Semp\,(M/N)$ such that $\phi\,(P) = P/N$. It is clear that ϕ is a bijective morphism. Now let $P,\ P' \in Semp\,(M)$, then $\phi\,(P \wedge P') = \phi\,(P \cap P') = (P \cap P')/N = (P/N) \cap (P'/N) = \phi\,(P) \wedge \phi\,(P')$. On the other hand, we notice that $\dfrac{\sqrt{P + P'}}{N} = \sqrt{\dfrac{p}{N} + \dfrac{P'}{N}}$. Therefore $\phi\,(P \vee P') = \phi\,(\sqrt{P + P'}) = \phi\,(P) \vee \phi\,(P')$. Hence ϕ is a morphism of frames. Analogously we can prove that inverse ϕ^{-1} is a morphism of frames. So $[N, M] \cong Semp\,(M/N)$ as frames.

By Proposition 5.9 we note that when M is a quasi projective multiplication module and N is a semiprime submodule of M, then the frame $Semp\,(M/N)$ can be considered as a subframe of $Semp\,(M)$. So we have the following proposition:

Corollary 5.10. Let R be a ring and M a multiplication R-module. If M is a quasi projective module, then the subframe $[N, M]$ of $Semp\,(M)$ is a spatial frame for all semiprime submodules N of M.

Proof. It follows from Proposition 5.9 and Corollary 5.7.

Theorem 5.11. Let R be a commutative ring and M a faithful multiplication R-module and $QM \neq M$ for all maximal ideals Q of R. Then the topological spaces $Spec\,(R)$ and $Spec\,(M)$ are homeomorphic.

Proof. We consider the function $\varphi : Spec\,(R) \to Spec\,(M)$ such that $\varphi\,(I) = IM$. By Proposition 2.21 we know that IM is prime in M. Moreover by Corollary 1.11 we have that φ is injective. Now if $N \in Spec\,(M)$, then by Proposition 2.21 we know that $N = ann\,(M/N)$ M and $ann\,(M/N)$ is a prime ideal of R. Thus φ is an epimorphism. We will show that φ is continuous. Let $\mathcal{U}\,(N)$ an open set of the Zariski´s topology of $Spec\,(M)$. As M is a multiplication module and $\mathcal{U}\,(N) = \mathcal{U}\left(\sqrt{N}\right)$, then we can suppose that N is a semiprime module. Hence $N = IM$ with I is a semiprime ideal of R. We claim that $\varphi^{-1}\,(\mathcal{U}\,(N)) =$ $\varphi^{-1}\,(\mathcal{U}\,(IM)) = \{J \in Spec\,(R) \mid \varphi\,(J) \in \mathcal{U}\,(IM)\} =$

$\{J \in Spec\,(R) \mid JM \in \mathcal{U}\,(IM)\} = \{J \in Spec\,(R) \mid IM \nsubseteq JM\}$. Now by Proposition 1.8 we have that $\{J \in Spec\,(R) \mid I \nsubseteq J\} = \mathcal{U}\,(I)$. So $\varphi^{-1}\,(\mathcal{U}\,(N))$ is an open set of Zariski´s topology of $Spec\,(R)$. Analogously we show that if $\mathcal{U}\,(I)$ is an open set of $Spec\,(R)$, then $\varphi\,(\mathcal{U}\,(I))$ is an open set of $Spec\,(M)$. Therefore $Spec\,(R)$ and $Spec\,(M)$ are homeomorphic topological spaces.

Theorem 5.12. Let R be a commutative ring and M a faithful multiplication R-module and $QM \neq M$ for all maximal ideals Q of R. Then $Semp\,(R) \cong Semp\,(M)$ as frames.

Proof. We define $\varphi : Semp\,(R) \to Semp\,(M)$ such that $\varphi\,(I) = IM$. We claim that φ is an order isomorphism. In fact let $I_1, I_2 \in Semp\,(R)$ such that $I_1 \subseteq I_2$, then $I_1 M \subseteq I_2 M$. So $\varphi\,(I_1) \subseteq \varphi\,(I_2)$. Now if $\varphi\,(I_1) = \varphi\,(I_2)$, then $I_1 M = I_2 M$. So by Corollary 1.9 we have that $I_1 = I_2$. Thus φ is injective. Now let $N \in Semp\,(M)$. As M is a multiplication module then by Proposition 2.23 and Proposition 2.24 we have that $ann\,(M/N)\,M = N$ and $ann\,(M/N)$ is a semiprime ideal of R. So φ is surjective. Thus φ is bijective and $\varphi^{-1}\,(N) = ann\,(M/N)$. Now suppose that $N_1, N_2 \in Semp\,(M)$ such that $N_1 \subseteq N_2$, then $ann\,(M/N_1)\,M \subseteq ann\,(M/N_2)\,M$. So by Proposition 1.8 we have that $ann\,(M/N_1) \subseteq ann\,(M/N_2)$. Thus $\varphi^{-1}\,(N_1) \subseteq \varphi^{-1}\,(N_2)$. By [23, Chapter III Proposition 1.1] we have that φ is a lattice isomorphism. Hence φ is a frame isomorphism.

Corollary 5.13. Let R be a commutative ring and M a faithful multiplication R-module and $QM \neq M$ for all maximal ideals Q of R. Then there exists a bijective correspondence between $Spec\,(R)$ and $Spec\,(M)$.

Proof. Let P be a prime ideal of R. By Proposition 2.21 we have that PM is prime in M. Thus the restriction $\varphi_{|Spec(R)} : Spec\,(R) \to Spec\,(M)$ is injective. Now if N is prime in M, then by Proposition 2.21 we have that $ann\,(M/N)$ is a prime ideal of R. Moreover $N = ann\,(M/N)\,M$. Therefore $\varphi_{|Spec(R)}$ is surjective. So $\varphi_{|Spec(R)}$ is bijective.

From the Definition 1.1. we note that it is natural to consider the annihilator of a module. The next definition was given in [6].

Definition 5.14. Let M and K be R-modules. The annihilator of K in M is defined as:

$$Ann_M(K) = \cap \{Ker(f) \mid f \in Hom(M, K)\}$$

Notice that $Ann_M(K)$ is a fully invariant submodule of M and it is the greatest submodule of M such that $Ann_M(K)_M K = 0$.

In [17, Section 5] the authors define $\Psi(M)$ which is a frame given by condition on annihilators. They show that $\Psi(M)$ is a spatial frame. When M is a duo module (in particular a multiplication module) we have that

$$\Psi(M) = \{N \subseteq M \mid N + Ann_M(Rn) = M, \forall n \in N\}$$

In [17] the following is shown: 1) If $N \in \Psi(M)$, then $N^2 = N$ [Proposition 5.3]. 2) If $K, N \in \Psi(M)$, then $K \cap N = K_M N$ [Proposition 5.4], 3) If $\{N_i\}_{i \in J} \subseteq \Psi(M)$, then $\sum_{i \in J} N_i \in \Psi(M)$ [Proposition 5.5]. To prove those results the authors contend that $N_M \sum_{i \in J} K_i = \sum_{i \in J} N_M K_i$, happens when M is projective in $\sigma[M]$. But when M is a multiplication R-module (with R a commutative ring) by Remark 5.1 we do not need that hypothesis to prove the same results.

For the definition of a spatial frame, see [16, Quantales Definition 4.31].

Theorem 5.15. Let R be a commutative ring. If M is a multiplication module. Then $\Psi(M)$ is a spatial frame.

Proof. It is similar to the proof given in [17, Theorem 5.6].

Note that if R is a ring with a commutative multiplication of ideals and M is a R-multiplication module, then we have that $\Psi(M)$ is a spatial frame.

The classical Krull dimension of a module M

The classical Krull dimension of a poset $(X, \leq)$ was defined in [1]. We use the poset $(Spec(M), \subseteq)$ and we give the classical Krull dimension for an R-module M.

Set $Spec^{-1}(M) = \emptyset$, and for an ordinal $\alpha > -1$ define

$$Spec^{\alpha}(M) = \left\{ P \in Spec(M) \mid P \le Q \in Spec(M) \Rightarrow Q \in \bigcup_{\beta < \alpha} Spec^{\beta}(M) \right\}$$

If an ordinal α with $Spec^{\alpha}(M) = Spec(M)$ exists, then the smallest of such ordinals is called the classical Krull dimension of M; it is denoted by $cl.K\dim(M)$.

Notice that if M is a multiplication module, then by Remark 2.2 we have that M has maximal submodules which are prime submodules of M.

So $Spec^0(M) = \{P \in Spec(M) \mid P$ is a maximal submodule of $M\}$.

Remark 5.16. Let M be an R-module. Then by [1, Proposition 1.4] we have that M has classical Krull dimension if and only if the poset $(Spec(M), \subseteq)$ is noetherian.

Notice that if M is a noetherian R-module, then the poset $(Spec(M), \subseteq)$ is noetherian, therefore M has classical Krull dimension.

Theorem 5.17. Let R be a commutative ring and M a faithful multiplication R-module and $QM \ne M$ for all maximal ideals Q of R. Then R has classical Krull dimension if and only if M has classical Krull dimension. Moreover. $cl.K\dim(M) = cl.K\dim(R)$.

Proof. By Theorem 5.12 and Corollary 5.13 we have that $\varphi(P_1) \subseteq \varphi(P_2) \Leftrightarrow P_1 \subseteq P_2$ where P_1 and P_2 are prime ideals of R. Therefore $\varphi(Spec^{\alpha}(M)) = Spec^{\alpha}(R)$ for all ordinal α. Moreover φ is injective. Hence $cl.K\dim(M) = \delta \Leftrightarrow cl.K\dim(R) = \delta$.

The following definition was given in [13, Definition 3.26].

Definition 5.18. A topological space $(\mathbf{X}, \mathcal{T})$ is said to be noetherian if and only if every ascending (descending) chain of open (closed) subsets is stationary, equivalently if and only if every open subset is compact.

Proposition 5.19. Let R be a ring and M a multiplication R-module. Suppose that $\mathbf{Z}(M)$ is a noetherian topological space, then M has a classical Krull dimension.

Proof. If $P_1 \subseteq P_2 \subseteq ... \subseteq P_n.....$ is a chain in $Spec(M)$, then $\mathcal{U}(P_1) \subseteq \mathcal{U}(P_2) \subseteq ... \subseteq \mathcal{U}(P_n).....$ is a chain in $\mathbf{Z}(M)$. As the Zariski Topology is noetherian, then there exists a natural number k such that $\mathcal{U}(P_k) = \mathcal{U}(P_{k+1})$. Now if $P_k \subsetneq P_{k+1}$, then $P_{k+1} \in \mathcal{U}(P_k)$, a contradiction. Therefore $P_k = P_{k+1}$. So $(Spec(M), \subseteq)$ is a noetherian set. Thus M has classical Krull dimension.

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