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Extreme binary black holes in a physical representation

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Stationary axisymmetric binary systems of unequal co- and counter-rotating extreme Kerr black holes separated by a conical singularity are studied. Both solutions are identified as two three-parametric subfamilies of the Kinnersley–Chitre metric, and fully depicted by Komar parameters: the two masses M_1 and M_2, and a coordinate distance R, where the angular momenta J_1 and J_2 are functions of these parameters. Our physical representation allows us to identify clearly some limits and novel physical properties.

Subject Index E01, E30, E31

1. Introduction

The well-known Kinnersley–Chitre (KCH) five-parametric exact solution [1] represents the extreme limit case of the so-called double-Kerr–NUT (Newman–Tamburino–Unti) solution developed by Kramer and Neugebauer in 1980 [2], which allows one to treat the superposition of two massive rotating sources in general relativity. Both solutions permit the study of the dynamical interaction between two Kerr-type sources in stationary axisymmetric spacetimes by solving properly the corresponding axis conditions. In this respect, Yamazaki [3] found an asymptotically flat special member of the KCH metric through a specific parametrization that makes the NUT parameter vanish [4], which is identical to the Tomamitsu–Sato solution with distortion parameter $\delta = 2$ [5]. A few years ago, after following the ideas provided by Yamazaki [3] to eliminate the NUT parameter, Manko and Ruiz [6] solved for the first time in an analytical way the axis condition that disconnects the region between sources, with the main purpose being to describe co- and counter-rotating binary black hole (BH) systems separated by a conical singularity [7,8]; i.e, a massless strut related to the interaction force between sources which is a measure of their gravitational attraction as well as the spin–spin interaction. Even though the Manko–Ruiz representation of the KCH metric allows us to clarify some physical aspects related to unequal binary systems, the total Komar [9] mass M and total angular momentum J of the binary BH system contain complicated formulas in terms of dimensionless parameters, which could lead to erroneous interpretations at the moment of assigning numerical values to them. Therefore, it is important to review once again the KCH solution in order to express the metric of two-body systems of unequal co- and counter-rotating extreme BHs separated by a strut in a representation with a more physical aspect.

The main goal pursued in this paper is a rederivation of the two three–parametric subfamilies of the KCH metric concerning co/counter-rotating BHs considered earlier in Ref. [6], but with the principal characteristic that now both solutions will be given in terms of arbitrary physical Komar parameters:

the masses M_1 and M_2, as well as the coordinate distance R. We will obtain some well-known limits of the KCH solution and other dynamical aspects not considered before; in particular, those related to the merging process of interacting BHs.

The paper is organized as follows. In Sect. 2 we describe the KCH exact solution as well as the two approaches considered earlier in Refs. [3,6]; in particular, the path used by Manko and Ruiz to solve the axis conditions in order to describe interacting binary BHs by means of two three-parametric special members of the KCH metric. Later, in Sect. 3 we begin with a new, more suitable, five-parametric representation of the KCH solution with the main objective to solve once again the axis conditions and depict both metrics for interacting BHs in a more realistic physical representation. Concluding remarks can be found in Sect. 4.

2. The KCH exact solution

Stationary axisymmetric spacetimes are well defined with the Papapetrou metric [10]

$$ds^2 = f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] - f(dt - \omega d\varphi)^2, \tag{1}$$

and Einstein vacuum field equations can be reduced by means of Ernst's formalism [11] into a new complex equation,

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{\rho\rho} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{zz}) = 2(\mathcal{E}_\rho^2 + \mathcal{E}_z^2), \tag{2}$$

where a suffix ρ or z denotes partial differentiation. It follows that one can find the metric functions $f(\rho,z)$, $\omega(\rho,z)$, and $\gamma(\rho,z)$ of the line element Eq. (1) by solving the following equations:

$$\begin{aligned} f &= \mathrm{Re}(\mathcal{E}), \\ \omega_\rho &= -4\rho(\mathcal{E} + \bar{\mathcal{E}})^{-2}\mathrm{Im}(\mathcal{E}_z), \qquad \omega_z = 4\rho(\mathcal{E} + \bar{\mathcal{E}})^{-2}\mathrm{Im}(\mathcal{E}_\rho), \\ \gamma_\rho &= \rho(\mathcal{E} + \bar{\mathcal{E}})^{-2}\left(\mathcal{E}_\rho\bar{\mathcal{E}}_\rho - \mathcal{E}_z\bar{\mathcal{E}}_z\right), \qquad \gamma_z = 2\rho(\mathcal{E} + \bar{\mathcal{E}})^{-2}\mathrm{Re}(\mathcal{E}_\rho\bar{\mathcal{E}}_z), \end{aligned} \tag{3}$$

once we know an analytical solution for the non-linear Eq. (2). In this sense, the KCH solution solves Eq. (2) exactly; it is described by the complex potential $\mathcal{E}$ which is given by[1]

$$\begin{aligned} \mathcal{E} &= \frac{\Lambda - 2\Gamma}{\Lambda + 2\Gamma}, \\ \Lambda &= (\alpha^2 - \beta^2)(x^2 - y^2)^2 + p^2(x^4 - 1) + q^2(y^4 - 1) - 2i\alpha(x^2 + y^2 - 2x^2y^2) \\ &\quad - 2ipqxy(x^2 - y^2) - 2i\beta xy(x^2 + y^2 - 2), \\ \Gamma &= e^{-i\gamma_o}[px(x^2 - 1) + iqy(y^2 - 1) - i(p\alpha + iq\beta)x(x^2 - y^2) + i(p\beta + iq\alpha)y(x^2 - y^2)], \end{aligned} \tag{4}$$

where (x,y) are prolate spheroidal coordinates depicted as

$$x = \frac{r_+ + r_-}{2\kappa}, \quad y = \frac{r_+ - r_-}{2\kappa}, \qquad r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2}, \tag{5}$$

which are related to the cylindrical coordinates (ρ,z) by means of

$$\rho = \kappa\sqrt{(x^2 - 1)(1 - y^2)}, \qquad z = \kappa xy. \tag{6}$$

[1] Kinnersley and Chitre used the inverse function of $\mathcal{E}$ in their original paper [1], i.e., $\xi = \frac{1-\mathcal{E}}{1+\mathcal{E}} = \frac{2\Gamma}{\Lambda}$.

It is worthwhile mentioning that the above solution Eq. (4) contains the real parameters p, q, γ_o, α, β, and half of the separation distance between the sources, κ, where the first three obey the constraints

$$p^2+q^2=1, \qquad |e^{-i\gamma_o}|=1. \tag{7}$$

Taking into account $y=1$ and $x=z/\kappa$, the Ernst potential on the upper part of the symmetry axis adopts the form

$$\begin{aligned}
&\mathcal{E}(\rho=0,z)=\frac{e_+(z)}{e_-(z)},\\
&e_\pm(z)=(p^2+\alpha^2-\beta^2)z^2\mp 2\kappa[(p+q\beta-ip\alpha)e^{-i\gamma_o}\pm i(pq+\beta)]z\\
&\qquad +\kappa^2(p^2-\alpha^2+\beta^2+2i\alpha)\pm 2\kappa^2 e^{-i\gamma_o}(q\alpha-ip\beta),
\end{aligned} \tag{8}$$

from which the first Geroch–Hansen multipolar moments [12,13] can be explicitly computed once we apply the Fodor–Hoenselaers–Perjés procedure [14]; they read [6]

$$\begin{aligned}
&M=\frac{2\kappa(pP-pQ\alpha+qP\beta)}{p^2+\alpha^2-\beta^2}, \qquad J=M\left[\frac{(pq+\beta)M+\kappa(qQ\alpha+pP\beta)}{pP-pQ\alpha+qP\beta}-2J_0\right],\\
&J_0=-\frac{2\kappa(pQ+pP\alpha+qQ\beta)}{p^2+\alpha^2-\beta^2}, \qquad e^{-i\gamma_o}:=P-iQ,
\end{aligned} \tag{9}$$

where M and J represent the total mass and total angular momentum of the system, respectively. Also, J_0 is the NUT parameter.[2] Starting with the previous axis data, Ref. [6] provides the full KCH metric via Sibgatullin's method [15], which is written down in a closed analytical form by using Perjés' factor structure [16]; it reads

$$\begin{aligned}
&f=\frac{N}{D}, \qquad \omega=2J_0(y-1)+\frac{\kappa(y^2-1)F}{N}, \qquad e^{2\gamma}=\frac{N}{K_0^2(x^2-y^2)^4},\\
&N=\mu^2+(x^2-1)(y^2-1)\sigma^2, \qquad D=N+\mu\pi-(y^2-1)\sigma\tau, \qquad F=(x^2-1)\sigma\pi+\mu\tau,\\
&\mu=p^2(x^2-1)^2+q^2(y^2-1)^2+(\alpha^2-\beta^2)(x^2-y^2)^2,\\
&\sigma=2\left[pq(x^2-y^2)+\beta(x^2+y^2)-2\alpha xy\right],\\
&\pi=(4/K_0)\{K_0\left[pPx(x^2+1)+2x^2+qQy(y^2+1)\right]+2(pQ+pP\alpha+qQ\beta)\\
&\qquad\times\left[pqy(x^2-y^2)+\beta y(x^2+y^2)-2\alpha xy^2\right]-K_0(x^2-y^2)\left[(pQ\alpha-qP\beta)x+(qP\alpha-pQ\beta)y\right]\\
&\qquad-2(q^2\alpha^2+p^2\beta^2)(x^2-y^2)+4(pq+\beta)x(\beta x-\alpha y)\},\\
&\tau=(4/K_0)\{K_0x\left[(qQ\alpha+pP\beta)(x^2-y^2)+qP(y^2-1)\right]+(pQ+pP\alpha+qQ\beta)y\\
&\qquad\times\left[(p^2-\alpha^2+\beta^2)(x^2-y^2)+y^2-1\right]-pQK_0y(x^2-1)-2p(q\alpha^2-q\beta^2-p\beta)(x^2-y^2)\\
&\qquad+(pq+\beta)(y^2-1)\},\\
&K_0=p^2+\alpha^2-\beta^2.
\end{aligned} \tag{10}$$

[2] Reference [6] does not consider the contribution of the NUT parameter J_0 inside the total angular momentum, which means that the full KCH metric contains two semi-infinite singularities located up and down along the symmetry axis.

First of all, one should notice that the above metric is invariant under the change $\{p, q, P, Q, \alpha, \beta\} \rightarrow \{-p, -q, -P, -Q, \alpha, \beta\}$. Secondly, such a metric is not asymptotically flat at spatial infinity ($x \rightarrow \infty$, $|y| < 1$), because $f \rightarrow 1$, $\gamma \rightarrow 0$, and $\omega \rightarrow 2J_0(y-1)$. According to Bonnor's description [17] the NUT charge defines a semi-infinite singular source that makes an additional contribution to the total angular momentum; i.e., it represents a massless rotating rod located along the lower part of the symmetry axis, $y = -1$ or $\theta = \pi$, since $y = \cos\theta$ in Boyer–Lindquist coordinates (r, θ). So, bearing in mind that asymptotically flat spacetimes can be obtained from Eq. (10) when the NUT parameter J_0 is eliminated, there exist several possibilities to achieve such a task. On one hand, Yamazaki [3] proposed the solution

$$P = \frac{p + q\beta}{\sqrt{(p+q\beta)^2 + p^2\alpha^2}}, \qquad Q = -\frac{p\alpha}{\sqrt{(p+q\beta)^2 + p^2\alpha^2}}, \tag{11}$$

while on the other hand Manko and Ruiz [6] went beyond in considering the following solution:

$$\alpha = -\frac{Q(p+q\beta)}{pP}. \tag{12}$$

Due to the fact that the metric function ω on the middle region between the sources ($x = 1$, $y = z/\kappa$) acquires the form

$$\begin{aligned} &q\alpha\left[QK_0 - (2p+P)\alpha\right] + \beta\left[pPK_0 + (2p+P)q\beta - 1 + 2p^2\right] - pq(1+pP) \\ &- (pQ + pP\alpha + qQ\beta)(q^2 + \alpha^2 - \beta^2) = 0, \end{aligned} \tag{13}$$

one notices that Yamazaki's approach does not simplify the above condition, while the second proposal considered by Manko and Ruiz factorizes it as follows:

$$\left[(p^2 - Q^2)\beta^2 - pq(1 + pP + Q^2)\beta - p^2(1+pP)\right]\left[(p^2 - Q^2)\beta - pq(pP + Q^2)\right] = 0, \tag{14}$$

which may eventually lead us to the description of two-body systems of unequal co/counter-rotating BHs separated by a massless strut by choosing the first/second factor, respectively. In this direction, over all the parametrization of Ref. [6], the total mass M and total angular momentum J of the system were given in terms of dimensionless parameters $\{p, q, P, Q\}$, and therefore the analysis of the dynamics for such BH systems was mostly performed in a numerical way. For instance, the total mass M and total angular momentum J in the counter-rotating sector are obtainable from the second factor of Eq. (14) in combination with Eq. (12); they assume the form

$$M = \frac{2\kappa(pP + q^2)}{p^2 - q^2}, \qquad J = \frac{2\kappa^2 q\left[(1+2pP)^2 - (p+P)^2\right]}{p(p^2 - q^2)^2}. \tag{15}$$

To make matters worse, the situation is even more complicated in the co-rotating sector where, after using the first factor of Eq. (14) together with Eq. (12), one obtains

$$\begin{aligned} M &= \frac{\kappa\left[\pm q\Delta_o - p(1+p^2) - q^2 P\right]}{p(p^2 - q^2)}, \\ J &= \frac{\kappa M}{2p(p^2 - q^2)}\{\pm\Delta_o(2p^2P - 2p - P) + 2q(1 + p^2 + pP) - qP(p^2 - q^2)(p - P)\}, \\ \Delta_o &:= \sqrt{4p^2(1+pP) + q^2(p+P)^2}. \end{aligned} \tag{16}$$

This last point naturally motivates the present work to consider another more suitable parametrization which might invert the problem and establish a real physical representation of the KCH metric to describe interacting BHs in a more transparent form.

3. Extreme binary black holes in a physical representation

The problem of expressing the KCH metric with a more physical aspect can be tackled by first adopting a new representation for such a solution. In order to do so, we begin with a new suitable parametrization of the Ernst potential on the symmetry axis,

$$\mathcal{E}(\rho=0,z)=\frac{z^2-[M+i(\mathfrak{q}+2J_0)]z+\frac{2\Delta-R^2}{4}+\frac{\mathfrak{q}(P_1+P_2)}{2M}-2\mathfrak{q}J_0+i\left(P_1-\frac{2J_0(P_2+M\mathfrak{q})}{\mathfrak{q}}\right)}{z^2+(M-i\mathfrak{q})z+\frac{2\Delta-R^2}{4}-\frac{\mathfrak{q}(P_1+P_2)}{2M}+iP_2}, \quad (17)$$

$$\Delta:=M^2-\mathfrak{q}^2,$$

where the KCH solution now contains the five parameters $\{M,\mathfrak{q},R,P_1,P_2\}$ related to the set $\{p,q,\gamma_o,\alpha,\beta,\kappa\}$ via the expressions

$$\mathfrak{q}=\frac{2\kappa[p(q+Q)+pP\alpha+(1+qQ)\beta]}{p^2+\alpha^2-\beta^2}, \qquad R=2\kappa,$$
$$P_1=\frac{2\kappa^2[(1-qQ)\alpha-pP\beta]}{p^2+\alpha^2-\beta^2}+2\left(M+\frac{P_2}{\mathfrak{q}}\right)J_0, \qquad P_2=\frac{2\kappa^2[(1+qQ)\alpha+pP\beta]}{p^2+\alpha^2-\beta^2}, \quad (18)$$

with M and J_0 as expressed previously. The inverse relation between these two sets of parameters is completed if we construct once again the full KCH metric by using Perjés' factor structure in the same way as in Ref. [6], leading us to

$$p=\frac{M(\alpha_+\beta_+-\alpha_-\beta_-)}{\sqrt{(\alpha_+^2+M^2\beta_-^2)(\alpha_-^2+M^2\beta_+^2)}}, \qquad q=-\frac{\alpha_+\alpha_-+M^2\beta_+\beta_-}{\sqrt{(\alpha_+^2+M^2\beta_-^2)(\alpha_-^2+M^2\beta_+^2)}},$$
$$P=\frac{M(\alpha_+\beta_++\alpha_-\beta_-)}{\sqrt{(\alpha_+^2+M^2\beta_-^2)(\alpha_-^2+M^2\beta_+^2)}}, \qquad Q=\frac{\alpha_+\alpha_--M^2\beta_+\beta_-}{\sqrt{(\alpha_+^2+M^2\beta_-^2)(\alpha_-^2+M^2\beta_+^2)}}, \quad (19)$$
$$\beta\pm\alpha=\frac{2MR[\mathfrak{q}\alpha_\pm\pm M(R\pm M)\beta_\mp][q_o\mp 2M^2R(\alpha_\mp-\mathfrak{q}\beta_\pm)]}{(\alpha_-^2+M^2\beta_+^2)(\alpha_+^2+M^2\beta_-^2)}, \qquad \kappa=R/2,$$
$$q_o:=\alpha_+\alpha_-+M^2\beta_+\beta_-, \qquad \alpha_\pm:=M(\Delta\pm MR)-\mathfrak{q}(P_1+P_2), \qquad \beta_\pm:=2P_2\pm\mathfrak{q}R.$$

Additionally, the total angular momentum J, as well as the NUT charge J_0, in this new representation are reduced to

$$J=M\mathfrak{q}-\frac{P_1-P_2}{2}+\frac{J_0P_2}{\mathfrak{q}},$$
$$J_0=\frac{\mathfrak{q}}{2M}\left(\frac{\mathfrak{q}^2(P_1+P_2)^2-M^2\left[4P_1P_2-\Delta(R^2-\Delta)\right]}{\mathfrak{q}^2\left[M(R^2-\Delta)+\mathfrak{q}(2P_1+2P_2+M\mathfrak{q})\right]-M(2P_2+M\mathfrak{q})^2}\right). \quad (20)$$

With the main purpose of describing the interaction between two extreme BHs separated by a conical singularity [7,8], the first equation that eliminates J_0 is

$$\mathfrak{q}^2(P_1+P_2)^2-M^2\left[4P_1P_2-\Delta(R^2-\Delta)\right]=0, \quad (21)$$

while after developing a few non-trivial calculations one gets a simple quadratic expression for the axis condition $\omega(x=1,y=2z/R)=0$, that disconnects the region between the sources, namely

$$\mathfrak{q}\left[M^2(R^2+MR+\mathfrak{q}^2)(P_1-P_2)^2+(\Delta+MR)(\Delta-MR-R^2)(P_1+P_2)^2\right]$$
$$-M^2(R^2-\Delta)\left\{\left[M\mathfrak{q}^2+(R+M)(R^2+MR+\mathfrak{q}^2)\right](P_1-P_2)-M\mathfrak{q}(R+M)(R^2-\Delta)\right\}=0, \tag{22}$$

and it is not complicated to show that Eqs. (21) and (22) contain a trivial set of solutions, which explicitly are

$$P_{1,2}=\frac{\mp\Delta[R^2+MR+\mathfrak{q}^2]+\epsilon M\sqrt{\Delta(R^2+MR+\mathfrak{q}^2)^2+M^2\mathfrak{q}^2(R^2-\Delta)}}{2M\mathfrak{q}},$$
$$P_{1,2}=\frac{\mp\mathfrak{q}\Delta+\epsilon M\sqrt{(R+M)^2(R^2-\Delta)+\mathfrak{q}^2\Delta}}{2(R+M)},\qquad \epsilon=\pm1, \tag{23}$$

where the subscripts 1 and 2 are associated with the $-$ and $+$ signs, while the sign of ϵ refers to the location of the sources. In the remainder of this paper we use $\epsilon=1$; this means that the first/second source will be located up/down, respectively. The aforementioned Eq. (23) gives us two three-parametric subfamilies of the KCH metric that we are going to explore in the following subsections. Since we have solved the axis condition in the middle region between the sources, the total ADM mass [18] will be exactly the sum of both individual masses of the binary system, and thereby the BHs will be separated by a massless strut. It is worth mentioning that Eq. (22) was derived recently in Ref. [19] for the case of non-extreme sources.

3.1. *Co-rotating binary black holes*

Using the first solution of Eq. (23), it can be possible to describe a co-rotating two-body system of unequal Kerr sources separated by a massless strut as a three-parametric subclass of the KCH metric. By means of Perjés' representation [16], the Ernst potential $\mathcal{E}$ and the full metric are depicted by

$$\mathcal{E}=\frac{\Lambda-2\Gamma}{\Lambda+2\Gamma},\qquad f=\frac{N}{D},\qquad \omega=\frac{R(y^2-1)F}{2N},\qquad e^{2\gamma}=\frac{N}{\mathfrak{q}^4R^4(x^2-y^2)^4},$$
$$\Lambda=\mathfrak{q}^2(R^2-\Delta)(x^2-y^2)^2+\Delta[\mathfrak{q}^2(x^4-1)+(R+M)^2(y^4-1)]$$
$$+2i\mathfrak{q}\left\{xy\left[\Delta(R+M)(x^2-y^2)-M(MR+\Delta)(x^2+y^2-2)\right]-\delta_1(x^2+y^2-2x^2y^2)\right\},$$
$$\Gamma=\left(\frac{\mathfrak{q}(\Delta-MR-R^2)+i\delta_1}{MR[(R+M)^2+\mathfrak{q}^2]}\right)\{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]\left[\mathfrak{q}x(x^2-1)-i(R+M)y(y^2-1)\right]$$
$$-\mathfrak{q}\left\{M(\Delta+MR)\left[(R+M)x-i\mathfrak{q}y\right]-\delta_1\left[(R+M)y-i\mathfrak{q}x\right]\right\}(x^2-y^2)\},$$
$$N=\mu^2+(x^2-1)(y^2-1)\sigma^2,\quad D=N+\mu\pi-(1-y^2)\sigma\tau,\quad F=(x^2-1)\sigma\pi-\mu\tau,$$
$$\mu=\mathfrak{q}^2(R^2-\Delta)(x^2-y^2)^2+\Delta\left[\mathfrak{q}^2(x^2-1)^2+(R+M)^2(y^2-1)^2\right],$$
$$\sigma=2\mathfrak{q}\left\{\mathfrak{q}^2Rx^2+\left[2M(\Delta+MR)-\mathfrak{q}^2R\right]y^2-2\delta_1xy\right\},$$
$$\pi=\left(\frac{4}{MR}\right)\{\mathfrak{q}^2x\left[M^2R\left(R(x^2-y^2)+2Mx\right)+\Delta(\Delta-MR-R^2)(1+y^2)-4M\delta_1y\right]$$
$$+y\left[2M\left(\Delta(R+2M)(R^2+\mathfrak{q}^2)+M^4R\right)y+\delta_1\left(M(\Delta+MR)(1+y^2)-\mathfrak{q}^2R(1+x^2)\right)\right]\},$$
$$\tau=\left(\frac{4\mathfrak{q}\Delta}{MR}\right)\{-x\left[(R^2+MR+\mathfrak{q}^2)(Rx+2M)x+(R+M)(\Delta-MR-R^2)\right]$$
$$+(1-x)y\left[M(R^2-\Delta)y+\delta_1(1+x)\right]+M\left((R+M)^2+\mathfrak{q}^2\right)\},$$

$$\delta_1 := \epsilon\sqrt{\Delta(R^2+MR+\mathfrak{q}^2)^2+M^2\mathfrak{q}^2(R^2-\Delta)}. \tag{24}$$

It is feasible to prove from Eq. (19) that the above metric is obtainable from the KCH metric [1,6] after making the following changes in the real parameters:

$$p=\frac{\mathfrak{q}}{\sqrt{(R+M)^2+\mathfrak{q}^2}},\qquad q=-\frac{R+M}{\sqrt{(R+M)^2+\mathfrak{q}^2}},\qquad e^{-i\gamma_o}=\frac{\mathfrak{q}(\Delta-MR-R^2)+i\delta_1}{MR\sqrt{(R+M)^2+\mathfrak{q}^2}},$$
$$\alpha=\frac{\mathfrak{q}\delta_1}{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]},\qquad \beta=\frac{M\mathfrak{q}(\Delta+MR)}{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]}. \tag{25}$$

On the other hand, the Komar integrals [9] for each mass and angular momentum can be calculated through Tomimatsu's formulae [20]:

$$M_i=-\frac{1}{8\pi}\int_{H_i}\omega\,\mathrm{Im}(\mathcal{E}_z)d\varphi dz,\qquad J_i=-\frac{1}{8\pi}\int_{H_i}\omega\left(1+\frac{1}{2}\omega\,\mathrm{Im}(\mathcal{E}_z)\right)d\varphi dz, \tag{26}$$

where the integrals must be evaluated over the corresponding horizon H_i. Apparently it seems quite complicated to develop such a goal; nevertheless, the technical difficulty of finding the correct formulas for the Komar masses and angular momenta of the sources can be circumvented by taking into account a limit process after expanding the above expressions around the values taken by x and y on the regions surrounding each BH—for instance, if we are surrounding the upper BH, one can take into the computer code $x=1+\varepsilon, y=1$ in the region on the axis $z>R/2$, for $|\varepsilon|\ll 1$, but in the region $|z|<R/2$ we now put $x=1, y=1-\varepsilon$. A trivial calculation yields the expressions

$$M_{1,2}=\frac{M}{2}\mp\frac{\delta_1}{2(\Delta+MR)},$$
$$J_{1,2}=M_{1,2}\left[\frac{\mathfrak{q}}{2}-\frac{\Delta(R+M)(R^2+MR+\mathfrak{q}^2)\pm(\Delta+MR)\delta_1}{2M\mathfrak{q}(R^2-\Delta)}\right], \tag{27}$$

and it is easy to observe that $M=M_1+M_2$. Furthermore, the expression $J=J_1+J_2$ allows us to recover the aforementioned Eq. (20) for the total angular momentum in the absence of the NUT charge, namely

$$J=M\mathfrak{q}-\frac{P_1-P_2}{2}=M\mathfrak{q}+\frac{\Delta(R^2+MR+\mathfrak{q}^2)}{2M\mathfrak{q}}, \tag{28}$$

whereas the difference between the values of the masses yields the relation

$$M_2-M_1=\frac{\delta_1}{\Delta+MR}, \tag{29}$$

and after replacing the explicit form of δ_1 which is denoted in Eq. (24), eventually one arrives at a bicubic equation for solving

$$\mathfrak{q}^6+3\mathfrak{a}_1\mathfrak{q}^4+3\mathfrak{a}_2\mathfrak{q}^2+\mathfrak{a}_3=0,$$
$$\mathfrak{a}_1:=(1/3)[2R^2+2MR-2M^2+(M_1-M_2)^2],$$
$$\mathfrak{a}_2:=(1/3)(R+M)[(R-M)(R^2+2MR-M^2)-2M(M_1-M_2)^2],$$
$$\mathfrak{a}_3:=-M^2(R+M)^2\left[R^2-(M_1-M_2)^2\right], \tag{30}$$

whose explicit roots are given by

$$\mathfrak{q}^2_{(k)} = -\mathfrak{a}_1 + e^{i2\pi k/3}\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{1/3} + e^{-i2\pi k/3}\mathfrak{a}_o\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{-1/3},$$
$$\mathfrak{a}_o := \mathfrak{a}_1^2 - \mathfrak{a}_2, \qquad \mathfrak{b}_o := (1/2)\left[3\mathfrak{a}_1\mathfrak{a}_2 - \mathfrak{a}_3 - 2\mathfrak{a}_1^3\right], \quad k = 0, 1, 2. \tag{31}$$

In this particular case we choose $k = 0$ since it defines entirely a real parameter $\mathfrak{q}$ which starts and ends at the same value given by the total mass M, where the coordinate distance runs from $R = 0$ to $R \to \infty$. Substitution of this real solution into Eq. (27) permits us to demonstrate that during the merging process ($R = 0$) each individual angular momentum J_i is related to its corresponding mass M_i by means of [19]

$$\frac{J_1}{M_1^2} = 1 + \frac{M_2}{M_1}, \qquad \frac{J_2}{M_2^2} = 1 + \frac{M_1}{M_2}, \tag{32}$$

where such a process conceives a single extreme BH of mass $M = M_1 + M_2$ and angular momentum $J = J_1 + J_2$, exactly satisfying a well-known formula for extreme BHs [19]:

$$J = J_1 + J_2 = (M_1 + M_2)^2. \tag{33}$$

Moreover, when the sources are far away from each other, in the limit $R \to \infty$ the simple expressions for extreme BHs are recovered, namely

$$\frac{J_1}{M_1^2} = 1, \qquad \frac{J_2}{M_2^2} = 1. \tag{34}$$

All these features mentioned can be seen in Fig. 1. Regarding now the dynamical aspects of this co-rotating two-body system, the interaction force associated with the strut can be computed straightforwardly by using the formula [8,21], to obtain

$$\mathcal{F} = \frac{1}{4}(e^{-\gamma_s} - 1) = \frac{\Delta\left[\mathfrak{q}^2 - (R+M)^2\right]}{4\,(\Delta + MR)^2} \equiv \frac{M_1M_2[(R+M)^2 - \mathfrak{q}^2]}{(R^2 - \Delta)[(R+M)^2 + \mathfrak{q}^2]}, \tag{35}$$

where γ_s is the value of the metric function γ evaluated on the region of the conical singularity; i.e., $\gamma(x = 1)$. The strut prevents the BHs from falling onto each other; it means that as both horizons get closer and closer, the interaction force $\mathcal{F} \to \infty$. The minimal distance occurs when $R \to 0$, and for that case $\mathfrak{q} \to M$ [see Eq. (30) or Fig. 1(a)]. Let us now assume that the sources move away from each other; thus, in the limit $R \to \infty$ one gets the following expansion:

$$\mathcal{F} \simeq \frac{M_1M_2}{R^2}\left[1 - \frac{2M^2}{R^2} + \frac{4M(M_1^2 + 8M_1M_2 + M_2^2)}{R^3} + O\left(\frac{1}{R^4}\right)\right], \tag{36}$$

which matches with the formula already given by Dietz and Hoenselaers [22] once we add the condition for extreme co-rotating sources given by Eq. (34). The strut might be removed if we consider $|\mathfrak{q}| = R + M$ in the above formula Eq. (35). Nevertheless, as was demonstrated first by Hoenselaers [23], the absence of a strut might produce the appearance of naked singularities (ring singularities) off the axis, since at least one of the two masses will be negative even if the total mass of the system does not violate the positive mass theorem [24,25]. The last statement can be confirmed directly from Eq. (27).

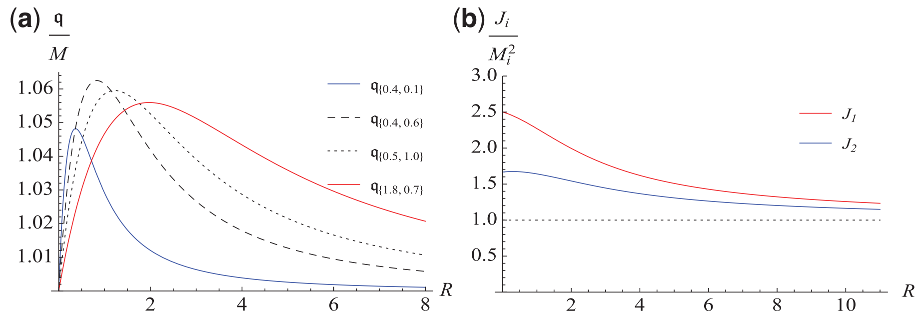

Fig. 1. (a) Behavior for the parameter $\mathfrak{q}$ in the co-rotating case taking different values for the masses M_1 and M_2 denoted by the subscripts inside the brackets, respectively. (b) The angular momenta J_1 and J_2 for the values $M_1 = 0.8$ and $M_2 = 1.2$.

To conclude the subsection, the identical case $M_1 = M_2 = m$, $J_1 = J_2 = j$ is recovered when the condition $\delta_1 = 0$ is imposed and a simple redefinition $\mathfrak{q} \rightarrow 2\mathfrak{q}$ is also taken into account; thus, one arrives at the extreme condition for identical co-rotating BHs, which was considered earlier in Refs. [26,27]:

$$m^2 - \mathfrak{q}^2 \left(1 - \frac{4m^2(R^2 - 4m^2 + 4\mathfrak{q}^2)}{[R(R+2m) + 4\mathfrak{q}^2]^2}\right) = 0, \tag{37}$$

where it can be shown that such a condition for identical extreme co-rotating BHs leads us to a bicubic equation, which is precisely the identical case of Eq. (30). Furthermore, after replacing such a condition in Eq. (28) [or Eq. (27)], the final expression for the equal angular momentum acquires the form [27]

$$j = \frac{m\mathfrak{q}[(R+2m)^2 + 4\mathfrak{q}^2]}{R(R+2m) + 4\mathfrak{q}^2}. \tag{38}$$

For identical constituents, the values for the angular momentum j are contained within the interval $1 < j/m^2 \leq 2$ [26], while for nonequal sources the ratio J_i/M_i^2 can be greater or lower than 2 [see Eq. (32) or Fig. 1(b)]. This peculiarity was first pointed out in Ref. [6] using a numerical argument.

3.2. *Counter-rotating binary black holes*

Regarding the second solution of Eq. (23), which refers to counter-rotating binary systems of unequal Kerr BHs also separated by a strut, where now the three-parametric member of the KCH exact solution is represented as follows:

$$\mathcal{E} = \frac{\Lambda - 2\Gamma}{\Lambda + 2\Gamma}, \qquad f = \frac{N}{D}, \qquad \omega = \frac{R(y^2-1)F}{2N}, \qquad e^{2\gamma} = \frac{N}{R^6(R+M)^4(x^2-y^2)^4},$$

$$\begin{aligned}\Lambda &= R\{(R+M)^2\left[(R^2-\Delta)(x^2-y^2)^2 + \Delta(x^4-1)\right] + \mathfrak{q}^2\Delta(y^4-1) \\ &\quad + 2i(R+M)\left(\mathfrak{q}xy\left[2\Delta(y^2-1) - R(R+M)(x^2+y^2-2)\right] - M\delta_2(x^2+y^2-2x^2y^2)\right)\},\\ \Gamma &= \left(\frac{\Delta + MR + i\delta_2}{(R+M)^2 + \mathfrak{q}^2}\right)\{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]\left[(R+M)x(x^2-1) + i\mathfrak{q}y(y^2-1)\right] \\ &\quad + (R+M)\left\{\mathfrak{q}(R^2+MR-\Delta)\left[\mathfrak{q}x + i(R+M)y\right] - M\delta_2\left[\mathfrak{q}y + i(R+M)x\right]\right\}(x^2-y^2)\},\end{aligned}$$

$$
\begin{aligned}
&N = \mu^2 + (x^2-1)(y^2-1)\sigma^2, \quad D = N + \mu\pi - (1-y^2)\sigma\tau, \quad F = (x^2-1)\sigma\pi - \mu\tau,\\
&\mu = R\left\{(R+M)^2\left[(R^2-\Delta)(x^2-y^2)^2 + \Delta(x^2-1)^2\right] + \mathfrak{q}^2\Delta(y^2-1)^2\right\},\\
&\sigma = 2R(R+M)\left[\mathfrak{q}R(R+M)(x^2+y^2) - 2\mathfrak{q}\Delta y^2 - 2M\delta_2 xy\right],\\
&\pi = (4/R)\{R(R+M)x\left[MR(R+M)\left(R(x^2-y^2)+2Mx\right) + \Delta(MR+\Delta)(1+y^2)\right] + \mathfrak{q}Ry\\
&\qquad \times\left\{2\mathfrak{q}y\left[R(R+M)^2 - \Delta(R+2M)\right] - \delta_2\left[(R+M)\left(R(x^2-y^2)+4Mx\right) + \Delta(1+y^2)\right]\right\}\},\\
&\tau = 4\Delta\{\mathfrak{q}\left\{(R+M)^2 + \mathfrak{q}^2 + x\left[MR+\Delta-(R+M)x(Rx+2M)\right] + (R^2-\Delta)(x-1)y^2\right\}\\
&\quad - \delta_2(R+M)y(x^2-1)\},\\
&\delta_2 := \epsilon\sqrt{(R+M)^2(R^2-\Delta) + \mathfrak{q}^2\Delta},
\end{aligned}
\tag{39}
$$

and this particular metric can be developed from the KCH metric [1,6] by means of

$$
\begin{aligned}
&p = \frac{R+M}{\sqrt{(R+M)^2+\mathfrak{q}^2}}, \qquad q = \frac{\mathfrak{q}}{\sqrt{(R+M)^2+\mathfrak{q}^2}}, \qquad e^{-i\gamma_o} = \frac{\Delta + MR + i\delta_2}{R\sqrt{(R+M)^2+\mathfrak{q}^2}},\\
&\alpha = \frac{M(R+M)\delta_2}{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]}, \qquad \beta = \frac{\mathfrak{q}(R+M)(R^2+MR-\Delta)}{\Delta\left[(R+M)^2+\mathfrak{q}^2\right]},
\end{aligned}
\tag{40}
$$

where we have substituted the second solution of Eq. (23) inside Eq. (19). The corresponding masses M_i and angular momenta J are given, respectively, by

$$
\begin{aligned}
M_{1,2} &= \frac{M}{2} \mp \frac{\mathfrak{q}(R^2+MR-\Delta)}{2\delta_2},\\
J_{1,2} &= M_{1,2}\left[\frac{\mathfrak{q}}{2}\left(2 - \frac{R^2}{R^2-\Delta}\right) \mp \frac{(\Delta+MR)\delta_2}{2(R+M)(R^2-\Delta)}\right],
\end{aligned}
\tag{41}
$$

where once again we have that $M = M_1 + M_2$. The expression of the total angular momentum $J = J_1 + J_2$ agrees with Eq. (20), acquiring the final form

$$
J = M\mathfrak{q} - \frac{P_1 - P_2}{2} = \mathfrak{q}\left(M + \frac{\Delta}{2(R+M)}\right), \tag{42}
$$

but now the difference between both masses gives

$$
M_2 - M_1 = \frac{\mathfrak{q}(R^2+MR-\Delta)}{\delta_2}, \tag{43}
$$

yielding another bicubic equation:

$$
\begin{aligned}
&\mathfrak{q}^6 + 3\mathfrak{b}_1\mathfrak{q}^4 + 3\mathfrak{b}_2\mathfrak{q}^2 + \mathfrak{b}_3 = 0,\\
&\mathfrak{b}_1 := (1/3)[2R^2 + 2MR - 2M^2 + (M_1-M_2)^2],\\
&\mathfrak{b}_2 := (1/3)[(R^2+MR-M^2)^2 - (M_1-M_2)^2(R^2+2MR+2M^2)],\\
&\mathfrak{b}_3 := -(M_1-M_2)^2(R+M)^2\left(R^2-M^2\right),
\end{aligned}
\tag{44}
$$

which has the roots

$$
\begin{aligned}
&\mathfrak{q}^2_{(k)} = -\mathfrak{b}_1 + e^{i2\pi k/3}\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{1/3} + e^{-i2\pi k/3}\mathfrak{a}_o\left[\mathfrak{b}_o + \sqrt{\mathfrak{b}_o^2 - \mathfrak{a}_o^3}\right]^{-1/3},\\
&\mathfrak{a}_o := \mathfrak{b}_1^2 - \mathfrak{b}_2, \qquad \mathfrak{b}_o := (1/2)\left[3\mathfrak{b}_1\mathfrak{b}_2 - \mathfrak{b}_3 - 2\mathfrak{b}_1^3\right], \quad k = 0,1,2.
\end{aligned}
\tag{45}
$$

Let us also consider the interaction force between the BHs, which now looks like

$$\mathcal{F} = \frac{\Delta[(R+M)^2 - \mathfrak{q}^2]}{4[(R+M)^2(R^2-\Delta) + \mathfrak{q}^2\Delta]} \equiv \frac{M_1M_2[(R+M)^2 - \mathfrak{q}^2]}{(R^2-\Delta)[(R+M)^2 + \mathfrak{q}^2]}. \tag{46}$$

Therefore, the expression of the force assumes an equivalent final form in both co/counter-rotating configurations of interacting BHs, but their dynamical and thermodynamical characteristics will differ considerably from each other at the moment of choosing values for $\mathfrak{q}$ that satisfy the cubic equation in each sector. The well-known identical counter-rotating BH systems are achieved by setting $\mathfrak{q} = 0$ in the above formulas of this subsection, from which one gets $M_1 = M_2 = m$ and $J_1 = -J_2 = -j$. Such configurations were first described analytically by Varzugin [28] after solving the corresponding Riemann–Hilbert problem; later on, Herdeiro et al. [29] provided several dynamical and thermodynamical aspects for these binary systems. In particular, they recognized the limit value $R \to 2m$ in which the merging process occurs, and the relation $|j| > m^2$ that violates the Kerr bound. In addition, Manko et al. [6,30] clearly identified the two-parametric subfamily member of the KCH metric that is recovered after setting $\mathfrak{q} = 0$ in Eq. (39). Last, but not least, Tomimatsu's equilibrium configurations without a supporting strut can be achieved when $\mathfrak{q} = R + M$, and $M = R(l-1)/(2-l)$ [20,23].

Continuing with the description and excluding the identical case, where now $\mathfrak{q} \neq 0$, we have noticed at least two possibilities in the relations between the masses given by the phase $k = 0$ in Eq. (45), where the coordinate distance R runs from $R = M_1 + M_2$ to $R \to \infty$. Without loss of generality, let us suppose that $M_2 > M_1$; in this regard $\mathfrak{q}$ acquires the final value $\mathfrak{q} = M_2 - M_1$ at infinity, while its initial value depends on the ratio between the masses at the moment that both sources are getting closer to each other. On one hand, if $M_2/M_1 < (3+\sqrt{5})/2 \simeq 2.61803$, the real parameter $\mathfrak{q}$ tends to a value close to zero, but never reaches it! Then, $\mathcal{F} \to \infty$ as M_2 approaches the value of M_1. On the other hand, if $M_2/M_1 > (3+\sqrt{5})/2$, $\mathfrak{q}$ takes an initial value given by

$$\mathfrak{q} = \sqrt{\left(\frac{M_2 - M_1}{2}\right)\left[\sqrt{25M^2 - 4M_1M_2} - (M_2 - M_1)\right] - M^2}, \tag{47}$$

but now the force remains finite. Fixing the mass $M_1 = 1$ and taking different values for the mass M_2 and the coordinate distance R, Table 1 provides several values for $\mathfrak{q}$, the angular momenta, and the force during the merging process. Some of these values are depicted in Fig. 2. Finally, when the sources are far away, the force behaves as

$$\mathcal{F} \simeq \frac{M_1M_2}{R^2}\left[1 - \frac{2(M_1^2 - 4M_1M_2 + M_2^2)}{R^2} + \frac{4(M_1-M_2)^2M}{R^3} + O\left(\frac{1}{R^4}\right)\right], \tag{48}$$

and thereby the result matches once more with the expression of Ref. [22], due to the fact that the individual angular momenta and masses satisfy the following relations at infinity:

$$\frac{J_1}{M_1^2} = -1, \qquad \frac{J_2}{M_2^2} = 1, \qquad J_1 < 0, J_2 > 0. \tag{49}$$

Table 1. Some numerical values for extreme counter-rotating BHs. The most violent merging process occurs at the limit value $R = 2m$ and it corresponds to the case of identical sources $M_1 = M_2 = m$, for which the interaction force $\mathcal{F} = \infty$ and each identical angular momentum $|j| = \infty$, in agreement with Ref. [29].

M_1	M_2	R	$\mathfrak{q}$	J_1	J_2	J	$\mathcal{F}$
1	1	2	0	$-\infty$	∞	0	∞
1	2	3.0001	0.0245	−367.439	367.531	0.0918	1667.17
1	2.618	3.6181	0.2411	−86.6621	87.7516	1.0895	44.3816
1	2.62	3.6201	0.2498	−83.7756	84.9047	1.1291	41.4199
1	3	4.0001	1.3038	−17.679	24.0594	6.3804	1.6727

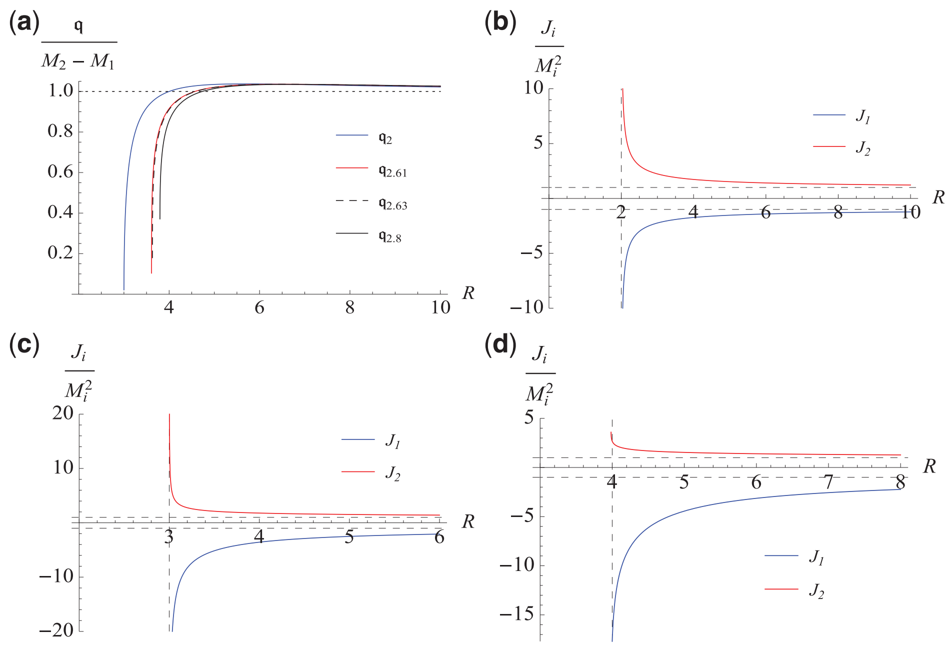

Fig. 2. (a) The parameter $\mathfrak{q}$ for counter-rotating BH systems fixing $M_1 = 1$ and assigning several values to the mass M_2 labeled by the subscripts. The angular momenta J_1 and J_2 for different values of M_2, where the merging limit is indicated by a vertical line given at the distance $R = M_1 + M_2$; for (b) $M_2 = 1$, (c) $M_2 = 2$, (d) $M_2 = 3$.

4. Concluding remarks

In the present paper we have worked out a concise physical representation for the two asymptotically flat three-parametric subfamilies of the KCH metric [1], that may be useful to describe in a more transparent way the interactions between co/counter-rotating binary BHs separated by a massless strut. In our opinion, this new parametrization is more suitable than the one presented in Ref. [6] when we want to describe the dynamical and physical properties of extreme binary systems; in particular, at the moment of choosing values for the masses and the separation distance. Additionally, our analysis has revealed that both descriptions of co/counter-rotating binary configurations are contained within the same formula for the interaction force, but their dynamical aspects differ from each other after solving a proper bicubic equation in each sector. These bicubic equations can be understood as dynamical laws for interacting BHs with struts and are special cases of the one previously obtained in Ref. [19]; it reads

$$\mathfrak{q}^3 - (a_1 + a_2)\mathfrak{q}^2 + (R + M)^2\mathfrak{q} - (R + M)[a_1(R + M_1 - M_2) + a_2(R - M_1 + M_2)] = 0, \quad (50)$$

with $a_i \equiv J_i/M_i$, $i = 1, 2$, being the angular momentum per unit mass. So, once we substitute the Komar parameters J_i of both co/counter-rotating two-body systems, their corresponding bicubic equations will emerge. Finally, we would like to point out that our physical representation of the KCH metric leads us to show clearly that the extreme solution saturates the Gabach Clement inequality [31]

$$\sqrt{1+4\mathcal{F}} = \frac{8\pi |J_i|}{S_i}, \tag{51}$$

where $\mathcal{F}$ is given by Eqs. (35) and (46), while S_i represents the area of the horizon S in the extreme limit case, obtainable after establishing $\sigma_i = 0$ in expression (36) of Ref. [19], having

$$S_i = 4\pi \frac{M_i^2[(R+M)^2 + \mathfrak{q}^2 - 2a_i\mathfrak{q}]^2 + a_i^2(R^2-\Delta)^2}{R^2[(R+M)^2+\mathfrak{q}^2]}, \tag{52}$$

and, therefore, it can be shown that equality is reached after placing the angular momenta on each rotating sector.

Acknowledgements

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