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ABSTRACT
The random effects in a gamma process are introduced in terms of its scale parameter. However, the scale parameter affects both its mean and variance. Hence, the variation of the degradation rates and the within degradation increments are expected to be large. For some products, the random effects affect just the rate or just the volatility of the process. Thus, two modifications of the parameters' structure of the gamma process are proposed. One implies that the random effects affect just the volatility and the second just the rate. A Bayesian estimation approach is provided and implemented in two case studies.

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1. Introduction

For some highly reliable products, its reliability assessment based on failure data may be complicated. The scarce obtaining of failure data makes challengeable the characterization of the failure time distribution and thus the general assessment of the product (Lu et al., 1996). Instead, it has been presented that the reliability assessment for most of the highly reliable products is based on the characteristics of the process that caused its failure. This process describes the accumulated damage over time and it is normally identified as a degradation process (Singpurwalla, 1995). In such cases, the reliability assessment consists in analyzing such degradation process until a critical level of degradation in which it is possible to characterize a failure of the product. Generally, these processes may describe the accumulated amount of wear, crack growth, corrosion, consumption, fatigue, contamination or the degradation of any performance characteristic (PC) of the product under normal use conditions or under accelerated degradation tests (ADT).

Degradation models based on gamma process have been identified as the main way to model degradation processes given the characteristic that its increments are independent and non-negative having a gamma distribution with an identical scale parameter. In the case of gamma degradation modeling, performance can only decrease over time, which makes it quite usable. (Park and Padgett, 2005; Bagdonavicius and Nikulin, 2000; Lawless and

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Crowder, 2004). Useful information about gamma process can be found in Noortwijk (2009), Bagdonavicius and Nikulin (2001), and Sinpurwalla (1995).

However, in real applications, it can be found that the degradation process of a product's characteristic is affected by different sources of variation. These are described by the inherent variation in the degradation for every specific product under study and the different behaviors of the degradation paths for every product. This implies that the degradation in a product population has a large variation due to some unobservable effects. The simple gamma process is unable to capture such variations. However, in the literature, these variations have been well modeled by incorporating random effects into the gamma process. The most used gamma process with random effects involves that the scale parameter of the process follows a certain distribution and it is unit-specific, such that it captures the different variations among products. Lawless and Crowder (2001) presented a gamma process with random effects in which it is considered that the different products in the populations have different realizations of the scale parameter while they share the same shape parameter. For this, they let the scale parameter of every products' degradation path follow a gamma distribution. It was found that the joint distribution of the scale parameter and the gamma process has a closed form in terms of a Fisher distribution, which makes straightforward the estimation of the parameters. This model has been widely used and studied in different applications (Tsai et al., 2012; Hao et al., 2015; Wang, 2008; Wang et al., 2015; Pulcini, 2013).

The random effects gamma process model described above considers that the scale parameter follows a gamma distribution. As the mean and variance of the gamma process are defined as a/b and a/b^2, respectively, where a is the shape parameter and b is the scale parameter. It can be noted that both the mean and variance are affected by the random effects parameter. This means that a unit with a high degradation rate, i.e., either a large a or a small b, will also have a high volatility. This implies that the inherent variation in the degradation for every specific product under study is expected to be large and that the behavior of the different degradation paths for every product tend to be quite different. However, for certain products it may be expected that the random effects affect only, either the degradation rate or the volatility of the process. Such may be the case for products which are characterized by a high level of degradation rate but a small variation of the within degradation increments of every product, or on the other case, products which are characterized by a large variation of the within degradation increments of every product and a low degradation rate. In this paper, different modifications of the structure of the parameters of the gamma process with random effects are considered in order to model the scenarios described above. As in the case of the classical gamma process with random effects, it is considered that the scale parameter of the process follows a gamma distribution. A time scale transformation is used in order to assure that degradation is a linear function of time (Whitmore and Schenkelberg, 1997). As the joint distributions of the proposed random effects models are complex, the estimation of the parameters is performed via Gibbs sampling and Markov chain Monte Carlo (MCMC) method by using the OpenBUGS software. The models are illustrated with the degradation modeling of two case studies based on the laser GaAs degradation dataset and a dataset of fatigue-crack growth. In addition, the proposed models are compared with the inverse Gaussian (IG) process models with random effects.

The rest of the paper is organized as follows. In Section 2, the simple gamma process and some important characteristics are presented. In Section 3, the classical gamma process with random effects and the proposed random effects models are presented. In Section 4, the inference method based on a Bayesian approach is described for the simple gamma process and the different models with random effects. In Section 5, the IG process is introduced and several modifications of the process are presented to introduce random effects. In Section 6, the illustrative examples are presented in order to illustrate the proposed models. In Section 7, the concluding remarks and discussion are presented.

2. Simple gamma process

Important characteristics of the gamma process are that it is a stochastic process with independent and non-negative increments. Considering a non-negative-valued process $\{Z(t),\ t > 0\}$, where $Z(t)$ represents the measured degradation for an individual unit at time t, then the gamma process has the following properties

- $Z(t) - Z(s) = \Delta Z(s, t)$ follows a gamma distribution $Ga(\alpha[\tau(t) - \tau(s)], u)$.
- $Z(t)$ has independent increments, $Z(t_4) - Z(t_3)$ and $Z(t_2) - Z(t_1)$ are independent $\forall t_1 < t_2 < t_3 < t_4$.

Now, let $\alpha\tau(t, \gamma)$ be a non-negative shape parameter with a time scale transformation in the form of $\tau(t, \gamma) = t^{\gamma}$, thus $[\tau(t, \gamma) - \tau(s, \gamma)] = \Delta\tau(s, t, \gamma) = t^{\gamma} - s^{\gamma}, t \geq 0, \upsilon(0) \equiv 0$, and $u > 0$ be an inverse scale parameter. Then $Z(t),\ t > 0$ is governed by a gamma process with the parameters described above.

Suppose that an increasing stochastic process describes the degradation level of some PC at time t, and it is governed by $Ga(\alpha\Delta\tau(s, t, \gamma), u)$ with mean $\alpha\Delta\tau(s, t, \gamma)/u$ and variance $\alpha\Delta\tau(s, t, \gamma)/u^2$. The moment of a failure caused by degradation is the moment when the degradation path reaches a critical level ω. Thus, the lifetime is defined as $T_{\omega} = \inf\{Z(t) \geq \omega\}$. The cumulative distribution function (CDF) of T_{ω} can be obtained as $P(Z(t) \geq \omega) = 1 - F_{\mathcal{G}a}(\omega, \alpha\tau(t_{\omega}, \gamma), u)$.

Considering a degradation test (DT) with the next characteristics: N units are tested and M measurements for all the units are observed up to the termination time T, which results in degradation measurements $Z_i(t_j)$ of the ith unit at the corresponding time t_j, $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M$. According to the independent increment property of the gamma process, and $\Delta Z_i(t_{j-1}, t_j) = Z_i(t_j) - Z_i(t_{j-1}),\ t_0 = 0,\ \Delta\tau_j(\gamma) = t_j^{\gamma} - t_{j-1}^{\gamma}$, for $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M$. Thus, it is possible to obtain independent random variables $\Delta Z_i(t_{j-1}, t_j) \sim Ga(\alpha\Delta\tau_j(\gamma),\ u)$, with CDF defined as $F_{\mathcal{G}a}(\Delta Z_i(t_{j-1}, t_j))$. this process is considered as a simple gamma process (SGP).

3. Gamma process with random effects

Considering that each product under study may experience different sources of variations during its operation, for a degradation model to be realistic, it is more appropriate to incorporate product-to-product variability in the modeling of the degradation process. In this paper, it is assumed that γ and υ are fixed parameters that are common to all products and u is a random parameter representing the heterogeneity among different products, this model is known as a gamma process with random effects (RE). Given that the mean $\alpha\Delta\tau_j(\gamma)/u$ and variance $\alpha\Delta\tau_j(\gamma)/u^2$ of the RE model are affected by the random effects parameter u. It is expected that; the degradation rate of the degradation paths tends to have a large dispersion. In addition, it is also expected that the variance of the degradation observations within each unit tends to be large. As the RE modeling is a general approach. It may be case that the set of products under study present only a large dispersion of the degradation rates, or only a large variation of the variance of the degradation observations within each unit. For such cases, different configurations of the parameters of the gamma process are proposed, such that the random effect parameter affect solely the mean or the variance of the degradation process. The reparametrizations consider the pair of parameters (υ, u) instead of (α, u), where $\alpha = \alpha(\upsilon, u) = \upsilon, \upsilon u, \upsilon u^2$. So, by considering that u is a random parameter following a gamma distribution as $u \sim Ga(\delta, \varphi)$ with mean δ/φ and variance δ/φ^2 and $\alpha(\upsilon, u) = \upsilon$, the classical gamma process with RE is obtained, if $\alpha(\upsilon, u) = \upsilon u$ it can be noted that only the variance is affected by u, this model is considered as a random variance model (RV), and if $\alpha(\upsilon, u) = \upsilon u^2$ only the mean is affected by u, this model is considered as a random mean model (RM). So, for all models the PDF of $\Delta Z_i(t_{j-1}, t_j)$ will be defined as

$$f(\Delta Z_i(t_{j-1}, t_j)) = \int_0^\infty f_{\mathcal{G}a}(\Delta Z_i(t_{j-1}, t_j)|\alpha(\upsilon, u)\Delta\tau_j(\gamma), u) f_{\mathcal{G}a}(u|\delta, \varphi)du \tag{1}$$

Then, the CDF of the lifetime when the degradation path reaches a critical level ω is defined as $t_\omega = \inf\{Z(t) \geq \omega\}$. The CDF of t_ω can be obtained as

$$F(t_\omega) = \int_0^\infty F_{\mathcal{G}a}(t_\omega|\alpha(\upsilon, u)\tau(t_\omega, \gamma), u) f_{\mathcal{G}a}(u|\delta, \varphi)du \tag{2}$$

It should be noted that when $\alpha(\upsilon, u) = \upsilon$, the models (1) and (2) of the RE model can be written in terms of the Fisher distribution (Lawless and Crowder, 2004) as, respectively,

$$F(\Delta Z_i(t_{j-1}, t_j)) = F_{2\upsilon\Delta\tau_j(\gamma),\ 2\delta}\left(\frac{\delta\Delta Z_i(t_{j-1}, t_j)}{\upsilon\varphi\Delta\tau_j(\gamma)}\right) \tag{3}$$

and,

$$F(t_\omega) = 1 - F_{2\upsilon\tau(t_\omega,\gamma),\ 2\delta}\left(\frac{\delta\omega}{\varphi\upsilon\tau\ (t_\omega, \gamma)}\right) \tag{4}$$

It is expected that in the RV model the variance of the degradation observations within each unit is significant, given that it is affected by the random effects parameter u. However, a low level of variation in the degradation rates may be observed, given that the degradation mean is not affected by the random effect parameter. Thus, the RV model is suitable for the degradation modeling of products for which overall degradation rate is low and a large unit-specific degradation variation exists. On the other hand, the mean function of the RM gamma process model varies to a certain level. It leads to a larger dispersion of the degradation rate, which is reflected in a larger variation of the first-passage time distribution. Thus, this model is appropriate for the modeling of the degradation of products in which a significant variation of the degradation rate within the products' sample is observed.

4. Parameters estimation

In this section, a general Bayesian approach is considered to estimate the parameters of interest of the models described in previous sections. In all cases, non-informative prior distributions are considered for the parameters. The Bayesian analysis is presented in the next order, first for the SGP model, then RE, and finally the RV and the RM model.

4.1. Bayesian analysis for SGP

Considering that degradation measurements $Z_i(t_j)$ have been observed of the ith unit at the corresponding time t_j, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Thus, $\Delta Z_i(t_{j-1}, t_j) = Z_i(t_j) - Z_i(t_{j-1})$ is the degradation increment in the interval time (t_{j-1}, t_j). Then, $\Delta Z_i(t_{j-1},\ t_j)$ are independent random variables that follow a gamma distribution with

$Ga(\alpha \Delta\tau_j(\gamma), u)$. Based on this, the Bayesian analysis is considered as follows. Non-informative prior distributions are considered for the unknown parameters. For υ a non-informative gamma distribution is considered as $Ga(\alpha, \beta)$. For u, a non-informative gamma distribution is considered as $Ga(\vartheta, \epsilon)$, and for γ a gamma distribution is also considered as $Ga(a_\gamma, b_\gamma)$. Where, $Ga(\alpha, \beta)$ is a gamma distribution with shape parameter α and scale parameter β, $Ga(\vartheta, \tau)$ is a gamma distribution with shape parameter ϑ and scale parameter ϵ, and $Ga(a_\gamma, b_\gamma)$ is also a gamma distribution with shape parameter a_γ and scale parameter b_γ. The prior distributions for υ and u were defined as that, because in the case of u it represents its conjugate prior distribution. In addition, the gamma distribution is considered to model u in order to introduce random effects in the SGP model. The prior distribution for υ was defined as a simple approximation to its conjugate prior distribution. In the case of γ, a gamma prior distribution is defined in order to avoid negative values of the time scale transformation.

The likelihood function for the degradation data under the gamma process model is defined as

$$L_{SGP}(\Delta Z_i(t_{j-1}, t_j)|\upsilon, u,\ \gamma) = \prod_{i=1}^{N}\prod_{j=1}^{M} f_{Ga}(\Delta Z_i(t_{j-1}, t_j)|\upsilon, \gamma, u) \tag{5}$$

Considering the prior distributions in (14) and the likelihood function in (15), the joint posterior distributions is obtained as follows

$$f_{SGP}(\upsilon, u,\ \gamma|\Delta Z_i(t_{j-1}, t_j)) \propto \pi(\upsilon\ |\alpha, \beta)\pi(u|\vartheta, \epsilon)\pi(\gamma\ |a_\gamma, b_\gamma) \times L_{SGP}(\Delta Z_i(t_{j-1}, t_j)|\upsilon, u, \gamma) \tag{6}$$

As can be seen, the joint posterior distribution results in a complex form. In order to estimate the parameters of interest $\theta_{SGP} = (\upsilon, u,\ \gamma)$ for SGP, the MCMC can be utilized to generate samples from the joint distribution. For such purpose, OpenBUGS software is used to implement the MCMC based on the Gibbs sampler. Further information about the implementation of the Gibbs sampler can be found in Gelfand and Smith (1990), Casella and George (1992), Smith and Roberts (1993), and Gelman et al. (2009).

4.2. Bayesian analysis for random effects models

Considering that degradation increments $\Delta Z_i(t_{j-1}, t_j)$ have been observed and that these increments follow a gamma distribution $Ga(\alpha(\upsilon, u)\Delta\tau_j(\gamma), u)$, with random effect parameter u that follows a gamma distribution $Ga(u\ |\delta, \varphi)$. Thus, the prior distributions are defined as follows. The prior distribution for υ is a gamma distribution $Ga(\alpha, \beta)$. The prior distribution for δ is a gamma distribution $Ga(a_\delta, b_\delta)$. The prior distribution for φ is a gamma distribution $\varphi \sim Ga(a_\varphi, b_\varphi)$, and the prior distribution for γ is also a gamma distribution $Ga(a_\gamma, b_\gamma)$.

Given that the random effects are described by the parameter u, each degradation path possesses a specific parameter u_i that follows a gamma distribution $Ga(\delta, \varphi)$ with $i = 1, 2, \ldots, N$. This implies that in all degradation paths $i = 1, 2, \ldots, N$, u_i follows the same random effects distributions $Ga(\delta, \varphi)$, which accounts for the pooling of the random effects information (Peng et al., 2014).

Considering that, the degradation paths $i = 1, 2, \ldots, N$ have the random effects u_i, and that all u_i follow the same gamma distribution $Ga(u_i|\delta, \varphi)$, thus the likelihood function can be described as

$$L_m(\Delta Z_i(t_{j-1}, t_j), u_i|\upsilon, \delta, \varphi, \gamma) = \prod_{i=1}^{N}\left\{ f_{\mathcal{G}a}(u_i|\delta, \varphi)\prod_{j=1}^{M} f_{\mathcal{G}a}(\Delta Z_i(t_{j-1}, t_j)|\alpha(\upsilon, u)\Delta\tau_j(\gamma), u_i)\right\} \quad (7)$$

where $m = RE, RV, RM$ depending in the form of $\alpha(\upsilon, u)$.

The joint posterior distribution for the RE model is defined as follows

$$f_m(\upsilon, \delta, \varphi, \gamma, u_i|\Delta Z_i(t_{j-1}, t_j)) \propto \pi(\upsilon|\alpha, \beta)\pi(\delta|a_\delta, b_\delta)\pi(\varphi|a_\varphi, b_\varphi)\pi(\gamma|a_\gamma, b_\gamma)L_m(\Delta Z_i(t_{j-1}, t_j), u_i|\upsilon, \delta, \varphi, \gamma) \quad (8)$$

As in the case of the SGP model, the parameters of interest $\theta_m = (\upsilon, \delta, \varphi, \gamma)$ are obtained via MCMC and the Gibbs sampler implemented in OpenBUGS. This by using the proposed scheme of prior distributions for every parameter. The code implemented in OpenBUGS for the three random effects models is slightly modified in the structure of the parameters of the RE gamma process as described in Section 3.

5. The IG process degradation model with random effects

The IG process as a degradation model has been found to be an attractive model to deal with degradation process with monotone degradation paths (Wang and Xu, 2010). This model has been thoroughly studied in the literature in various applications (Qin et al., 2013; Chen et al., 2015; Pan et al., 2016). Extensions of the simple IG process have also been introduced in the literature in the aims of dealing with degradation processes that are characterized by unexplained heterogeneous sources of variation. Ye and Chen (2014) introduced different IG models with random effects, specifically an IG model with random drift (IG-RD), a model with random volatility (IG-RV) and a third model known as random drift-volatility model (IG-RDV). Peng et al. (2014) introduced a modification of the IG-RD model and developed the Bayesian estimation scheme for the IG models with random effects. The IG process with drift parameter (μ) and diffusion parameter (λ) has the following characteristics: 1) $Z(t) - Z(s) = \Delta Z(t)$, follows an IG distribution $IG(\mu[\tau(t) - \tau(s)], \lambda[\tau(t) - \tau(s)]^2)$, 2) $Z(t)$ has independent increments, $Z(t_4) - Z(t_3)$ and $Z(t_2) - Z(t_1)$ are independent $\forall t_1 < t_2 < t_3 < t_4$. Where, $\tau(t)$ is a monotone increasing function. In this case, $\tau(t)$ is also considered as a monotone time-scale transformation in the form $\tau = t^\gamma$. Considering that, $\Delta Z(t)$ is governed by $IG(\mu\tau(t, \gamma), \lambda[\tau(t, \gamma)]^2)$ with mean $\mu\tau(t, \gamma)$ and variance $\mu^3\tau(t, \gamma)/\lambda$, and has the following PDF,

$$f_{\mathcal{IG}}(\Delta Z(t)) = \sqrt{\frac{\tau^2(t, \gamma)}{2\pi\Delta Z^3(t)}}\exp\left\{-\frac{(\Delta Z(t) - \mu\tau(t, \gamma))^2}{2\mu^2\Delta Z(t)}\right\} \quad (9)$$

Model in (9) is considered as a simple IG (SIG) model. Then, random effects are introduced by letting the parameters (μ, λ) follow certain distributions. For the IG-RD model, it is considered that μ^{-1} follows a truncated normal distribution $TN(A, \kappa^{-2})$. In this paper, for the IG-RD model the modification of the structure of the parameters of the IG process proposed by Peng et al. (2014) is considered, which consists in $\Delta Z(t) \sim IG(\mu\tau(t, \gamma), \lambda\mu^3[\tau(t, \gamma)]^2)$. For the IG-RV model the random effects are introduced by letting that λ follows a gamma distribution $Ga(B, \xi)$. While, for the IG-RDV model the random effects are introduced by letting that μ follows a truncated normal distribution $TN(A, \kappa^{-2})$ and by considering the modification of the structure of the parameters of the IG process as $\Delta Z(t) \sim IG(\mu\tau(t, \gamma), \lambda\mu^2[\tau(t, \gamma)]^2)$.

6. Illustrative examples

In this section, the SGP, RE, RV, and RM models are illustrated and compared with their implementation in two case studies. In addition, the gamma process models with random effects are compared with the IG models (SIG, IG-RD, IG-RV, IG-RDV). In the first case study, laser GaAs degradation dataset is used. While in the second case study fatigue-crack growth data is used.

6.1. Laser GaAs degradation case study

In this section, the laser data presented by Meeker and Escobar (1998) is used in order to illustrate the random effects models presented in previous sections. The degradation dataset describes the increase of operating current over time for 15 GaAs laser devices. When the operating current reaches a critical level of degradation, the device is considered to have failed. The sample size is $N = 15$, the observation times t_j, $j = 1, 2, \ldots, 16$, are the same for all the samples with $t_j = (250, 500, 750, \ldots, 4000)$ hours. In Fig. 1, the cumulative degradation paths are presented.

The performance of the proposed models in Sections 2 and 3 is compared. Considering, (a) the simple gamma process with $\Delta Z_i(t_{j-1}, t_j) \sim Ga(\alpha\Delta\tau_j(\gamma), u)$ and mean $\alpha\Delta\tau_j(\gamma)/u$ and variance $\alpha\Delta\tau_j(\gamma)/u^2$, (b) the RE model with $\Delta Z_i(t_{j-1}, t_j) \sim Ga(\upsilon\Delta\tau_j(\gamma), u)$, $u \sim Ga(\delta, \varphi)$, $u > 0$ and mean $\upsilon\Delta\tau_j(\gamma)/u$ and variance $\upsilon\Delta\tau_j(\gamma)/u^2$, (c) the RV model with $\Delta Z_i(t_{j-1}, t_j) \sim Ga(\upsilon\Delta\tau_j(\gamma)u, u)$, $u \sim Ga(\delta, \varphi)$, $u > 0$ and mean $\upsilon\Delta\tau_j(\gamma)$ and variance $\upsilon\Delta\tau_j(\gamma)/u$, (d) the RM model with $\Delta Z_i(t_{j-1}, t_j) \sim Ga(\upsilon\Delta\tau_j(\gamma)u^2, u)$, $u \sim Ga(\delta, \varphi)$, $u > 0$ with mean $\upsilon\Delta\tau_j(\gamma)u$ and variance $\upsilon\Delta\tau_j(\gamma)$. In addition, the Bayesian approach described

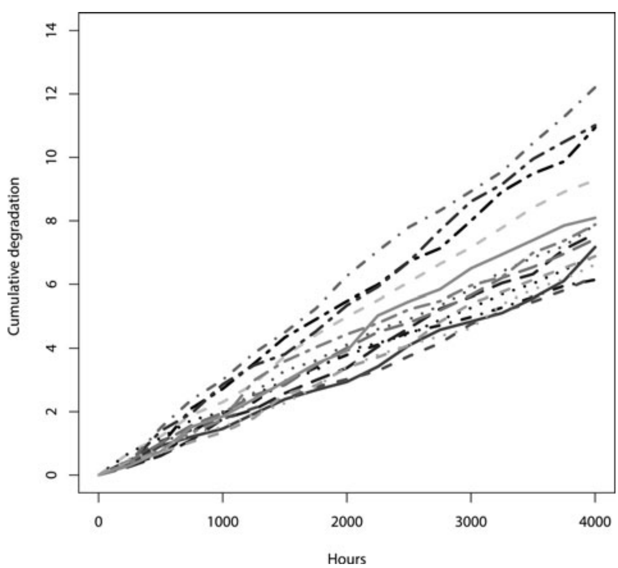

Figure 1. Cumulative degradation paths for GaAs laser degradation dataset.

Table 1. Estimations for the different gamma processes with random effect for the laser GaAs dataset.

	SGP model				RE model		
	mean	$p_{0.025}$	$p_{0.975}$		mean	$p_{0.025}$	$p_{0.975}$
υ	0.03011	0.01914	0.04437	υ	0.03965	0.02626	0.0553
u	14.25	11.86	16.94	γ	0.9988	0.96	1.039
γ	0.9979	0.9514	1.045	δ	40.95	16.08	82.37
				φ	2.111	0.788	4.403
	RV model				RM model		
	mean	$p_{0.025}$	$p_{0.975}$		mean	$p_{0.025}$	$p_{0.975}$
υ	0.002108	0.001395	0.003046	υ	1.04×10^{-4}	6.89×10^{-5}	1.51×10^{-4}
γ	0.9984	0.952	1.045	γ	1.011	0.9711	1.05
φ	89.81	10.65	198.4	φ	2.447	0.9249	5.249
δ	1226	146.9	2584	δ	44.66	17.91	91.83

in Sections 4.1 and 4.2 was implemented in OpenBUGS in order to obtain the estimations for $\theta_{SGP} = (\alpha, u, \gamma)$, $\theta_m = (\upsilon, \delta, \varphi, \gamma)$. Two sets of initial values are considered in the algorithm in order to assess the convergence of the parameters of interest with the Brooks-Gelman-Rubin (BGR) statistic (Gelman and Rubin, 1992). A total of 50,000 iterations were considered for burn-in, and 50,000 were considered for estimation purposes. As two sets of initial values were determined for every parameter, the BGR statistic was calculated for the parameters of interest. In general, it was found that convergence is achieved in every parameter. The obtained estimations and the respective percentiles of 2.5% and 97.5% are presented in Table 1.

Considering the estimations in Table 1, ten degradation paths of every random effect model were simulated in order to compare their behavior with the degradation paths of the laser GaAs dataset. In Fig. 2, the simulated degradation paths are presented. Information criteria were used in order to select the best fitting random effect model. The Deviance Information Criterion (DIC) was used for such purpose, which is an appealing tool for model selection based on information. The model with the lowest value of DIC is considered as the best fitting model (Spiegelhalter et al., 2002). The obtained DIC values are presented in Table 2.

As can be noted from Table 2 the random effect model with the smallest DIC value is the RM model. This may indicate that the RM model is the better fitting model for the laser GaAs dataset. However, the difference with the other models is slight. For instance, the RE model may be a good choice to model the GaAs laser degradation dataset. On the other hand, the model with the greatest DIC value is the SGP model. The gamma process degradation models were also compared to the IG models described in Section 5. The GaAs laser degradation dataset was also fitted to the SIG, IG-RD, IG-RV, and IG-RDV models. The corresponding DIC values were obtained as follows: for the SIG model a value of −144.8 was obtained, for the IG-RD a value of −176.7, for the IG-RV model a value of −138.6, for the IG-RDV model a value of −186.6. Among the IG models it can be noted that the DIC favors the IG-RDV model as in the work of Ye and Chen (2014). However, they suggested to use the IG-RD model for the GaAs laser degradation dataset, which is equivalent to the gamma RM model. All in all, it can be noted that the gamma models with RE and RM are favored above all the IG models. And, still the gamma RM model is favored by the DIC among all models.

For the assessment of the goodness-of-fit of the proposed gamma models and SGP and RE models, the Q-Q plotting may be considered as a qualitative tool to visually identify the best fitting model. The Q-Q plots are constructed by considering that when the underlining degradation process is governed by a gamma process, then

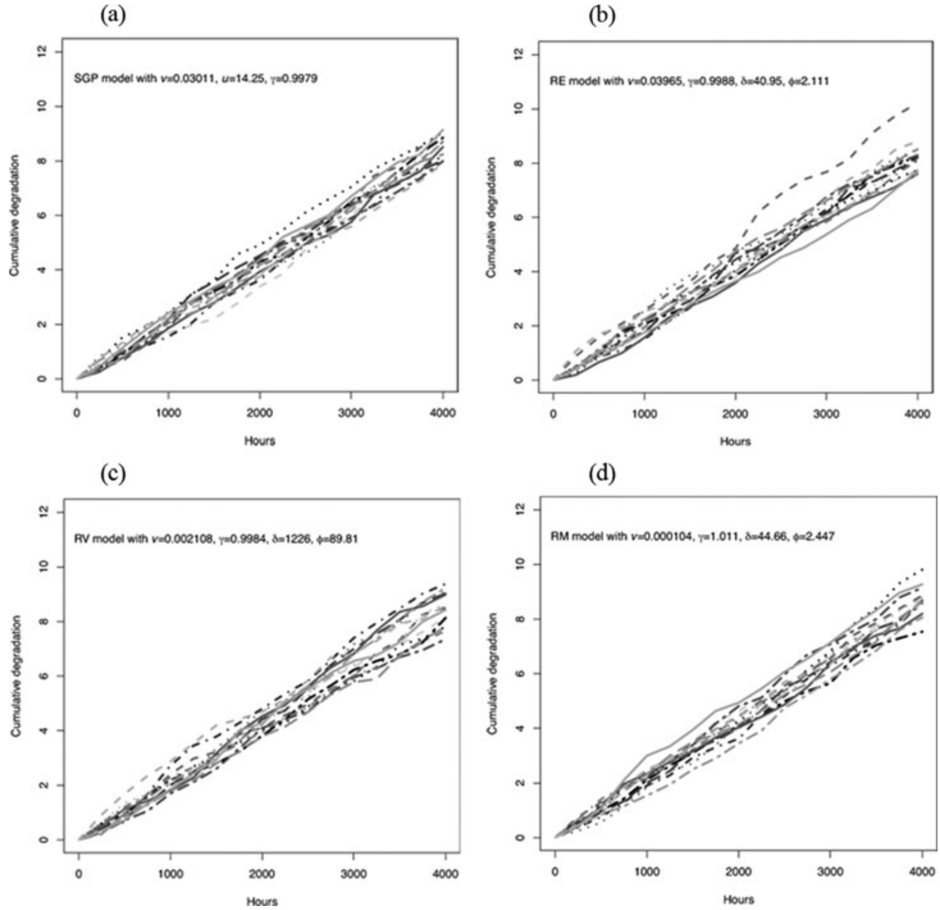

Figure 2. Simulated degradation paths with random effects based on the laser GaAs dataset: (a) SGP model, (b) RE model, (c) RV model, (d) RM model.

$\sqrt[3]{\hat{u}_i[Z_i(t_j) - Z_i(t_{j-1})]/[\hat{\alpha}(\hat{v}, \hat{u})(t_j) - \hat{\alpha}(\hat{v}, \hat{u})(t_{j-1})]}$ is approximately normal with estimators of the mean $1 - 1/[9\hat{\alpha}(\hat{v}, \hat{u})(t_j) - 9\hat{\alpha}(\hat{v}, \hat{u})(t_{j-1})]$ and variance $1/[9\hat{\alpha}(\hat{v}, \hat{u})(t_j) - 9\hat{\alpha}(\hat{v}, \hat{u})(t_{j-1})]$ (Wang and Xu, 2010; Ye et al., 2014). Where, $\hat{u}_i$ is an estimate of the random effects parameter for the ith unit, which per Lawless and Crowder (2004) can be $[\hat{\alpha}(\hat{v}, \hat{u})(t_j) + \hat{\delta}]/[Z_i(t_j) + \hat{\varphi}]$. By considering this, normal Q–Q plots can be constructed. In Fig. 3, the Q-Q plots for the different models are presented, it can be noted that the RM model has the best fitting to the data. In addition, the Anderson-Darling (AD) coefficient was computed for all the models, for the SGP model the AD was 0.703, for RE was 0.680, for RV was 0.997, for RM was 0.606. By considering the normal critical value of the AD statistic for a 0.1

Table 2. DIC values for gamma random effects models for laser GaAs dataset.

Model	DIC
SGP	− 133.9
RE	− 195
RV	− 134
RM	− 196

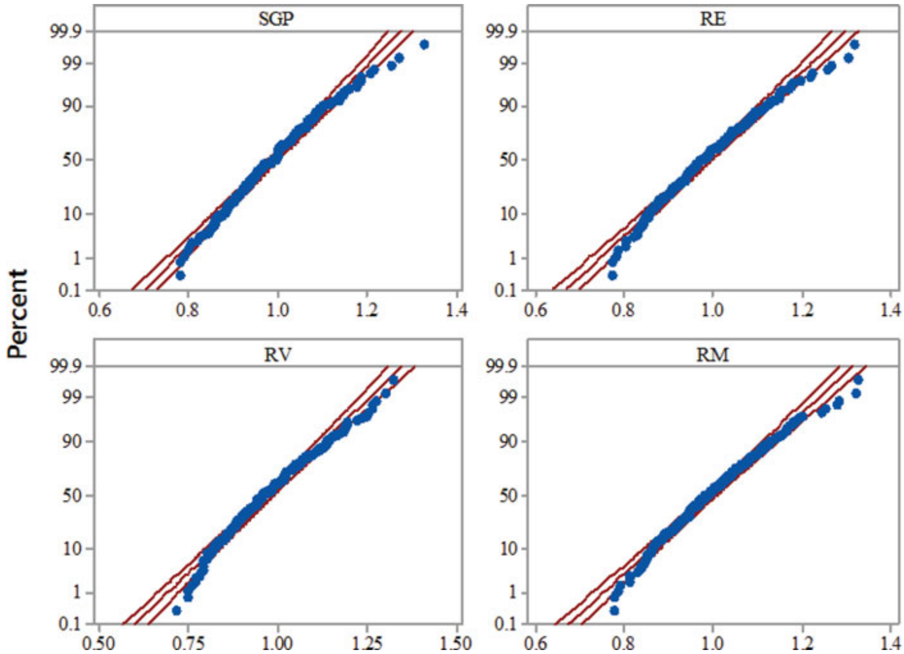

Figure 3. Comparison of the Q-Q plots for the different gamma process models for the laser GaAs data.

significance level of 0.63, it can be noted that the RM model is not rejected. As mentioned in previous sections, the RM model is suitable for the degradation modeling of products with a low level of variation of the degradation increments within each degradation path, and a large variation of the degradation rate, as can be seen in the degradation paths in Fig. 1.

6.2. Fatigue-crack growth case study

This case study consists in the crack propagation of a crack in a terminal of an electronic device. The function of the terminal is to transfer a signal to a receptor. The propagation of the crack to a certain critical length can lead to failure of the device given the inability of transferring the signal to the receptor. A DT was carried out in order to study the propagation of the crack in the terminal of ten devices. The crack growth for the terminal was measured every 0.1 hundred thousand cycles until 0.9 hundred thousand cycles. It is considered that the device has failed if the length of the crack exceeds the critical limit of 0.4 mm. The obtained

Table 3. Fatigue-crack growth increments dataset.

	Hundred thousands of cycles									
Device	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0.01	0.02	0.025	0.052	0.058	0.018	0.017	0.06	0.042
2	0	0.09	0.071	0.011	0.075	0.012	0.022	0.09	0.03	0.028
3	0	0.01	0.05	0.021	0.037	0.024	0.016	0.011	0.063	0.03
4	0	0.016	0.06	0.011	0.017	0.023	0.071	0.01	0.01	0.04
5	0	0.036	0.06	0.08	0.028	0.038	0.039	0.044	0.09	0.08
6	0	0.014	0.088	0.01	0.082	0.083	0.012	0.016	0.03	0.056
7	0	0.037	0.027	0.014	0.018	0.028	0.04	0.07	0.02	0.072
8	0	0.035	0.051	0.019	0.069	0.093	0.01	0.07	0.014	0.023
9	0	0.067	0.081	0.013	0.012	0.011	0.034	0.011	0.01	0.046
10	0	0.025	0.027	0.012	0.012	0.075	0.036	0.018	0.017	0.04

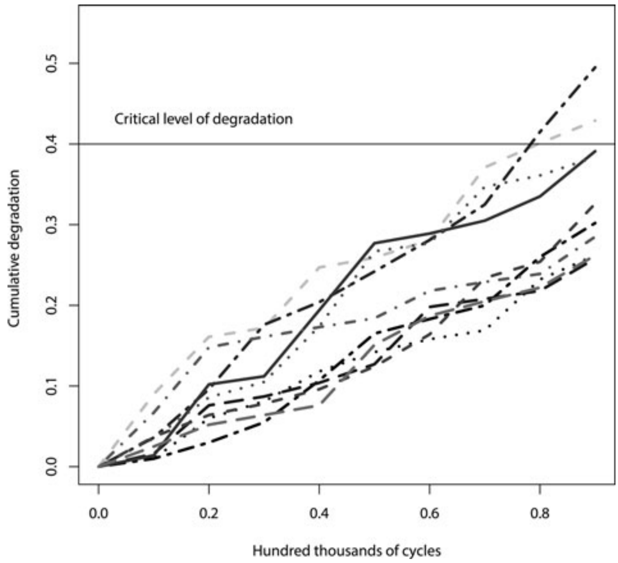

Figure 4. Cumulative degradation paths for fatigue-crack growth.

data is presented in Table 3; the units are in millimeters. In Fig. 4, the cumulative degradation paths are presented.

The proposed models presented in Section 3 are considered to model the fatigue-crack growth dataset. In addition, the Bayesian approaches described in Section 4 are also considered to estimate the parameters of interest from every model. An algorithm based on MCMC via Gibbs sampling was implemented in OpenBUGS in order to obtain the estimations for $\theta_{SGP} = (\alpha, u, \gamma)$, $\theta_m = (\upsilon, \delta, \varphi, \gamma)$. As in the previous case study, two sets of initial values are considered in the algorithm in order to assess the convergence of the parameters of interest with the BGR. A total of 50,000 iterations were considered for burn-in, and 50,000 were considered for estimation purposes. In general, it was found that convergence is achieved in every parameter according to the BGR graph obtained from OpenBUGS. The obtained estimations and the respective percentiles of 2.5% and 97.5% are presented in Table 4.

Considering the estimations in Table 4, ten degradation paths of every random effect model were simulated in order to compare their behavior with the degradation paths of the fatigue-crack growth dataset. In Fig. 5, the simulated degradation paths are compared. The DIC was

Table 4. Estimations for the different gamma processes with random effect for the fatigue-crack data.

	SGP model				RE model		
	mean	$p_{0.025}$	$p_{0.975}$		mean	$p_{0.025}$	$p_{0.975}$
υ	22.49	16.93	29.02	υ	21.86	16.63	28.47
u	59.81	43.38	79.1	γ	1.02	0.8913	1.163
γ	1.02	0.8929	1.161	δ	1566	315.6	3210
				φ	27.69	5.227	60.06
	RV model				RM model		
	mean	$p_{0.025}$	$p_{0.975}$		mean	$p_{0.025}$	$p_{0.975}$
υ	0.3846	0.3328	0.4465	υ	0.00772	0.004916	0.01171
γ	1.02	0.8897	1.164	γ	1.021	0.8879	1.168
φ	25.52	10.64	47.35	φ	44.49	10.31	99.52
δ	1388	534	2708	δ	2238	601.6	4769

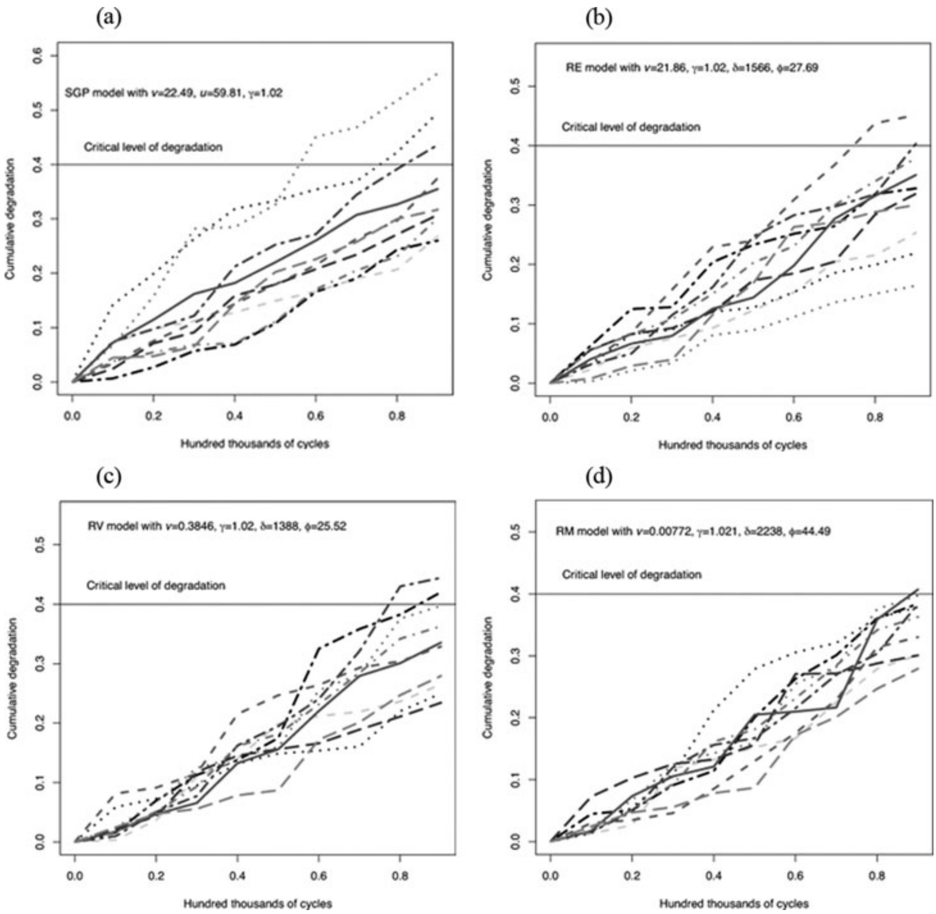

Figure 5. Simulated degradation paths with random effects based on the fatigue-crack growth dataset: (a) SGP model, (b) RE model, (c) RV model, (d) RM model.

also used to select the best fitting random effect model. The obtained DIC values are presented in Table 5.

As can be noted from Table 5 the random effect model with the smallest DIC value is the RE model. This may indicate that the RE model is the better fitting model for the fatigue-crack dataset. However, the difference of the DIC values among the different models is quite small. On the other hand, the model with the greatest DIC value is the SGP model. It can be noted that the variation of the degradation increments within degradation paths in the degradation paths of the fatigue-crack data in Fig. 4 is significant, but not as much as in the degradation paths in Figs. 5(a) and (c). It can also be noted that the level of variation of the degradation rate in the paths of Fig. 4 is low, but not as low as in the degradation paths in

Table 5. AIC values for random effects models for fatigue-crack growth dataset.

Model	AIC
SGP	− 429.8
RE	− 430
RV	− 429.6
RM	− 429.3

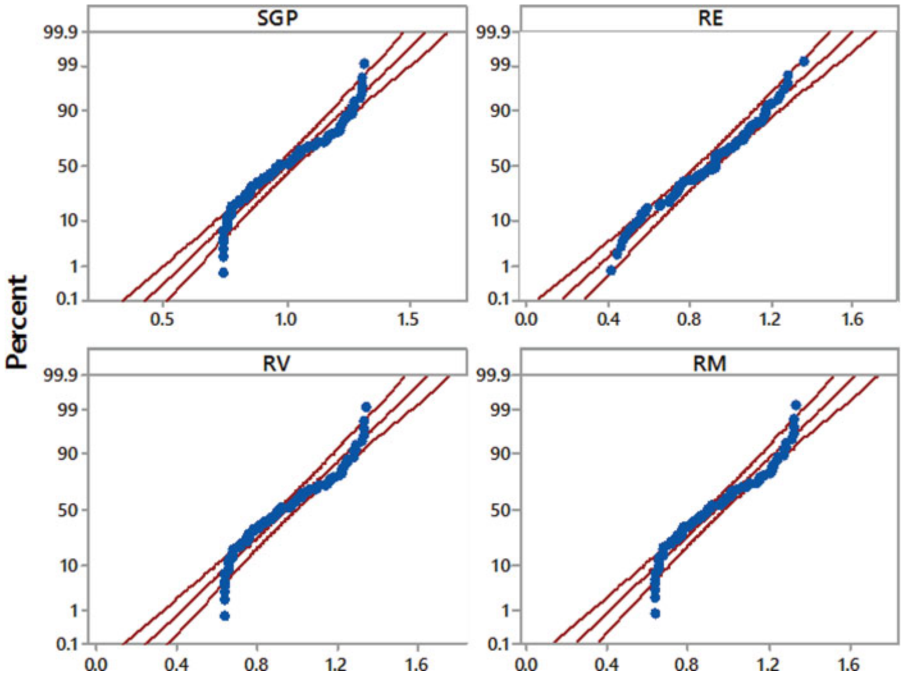

Figure 6. Comparison of the Q-Q plots for the different gamma process models for the fatigue-crack data.

Fig. 5(d). The behavior of the degradation paths in Figs. 4 and 5(b) are quite similar in terms of the variation of the degradation rate and the degradation increments within degradation paths. As in the laser degradation dataset, the fatigue-crack growth data was also fitted to the IG models, the DIC value was obtained as follows: for the SIG model the obtained value was −436.3, for the IG-RD model the value was −436, for the IG-RV model the value was −436.1, and for the IG-RDV model −436. Among the IG models, it can be noted that the DIC slightly favors the SIG model. In general, it can be noted that the IG models are favored above all the gamma models. So, for the fatigue-crack growth dataset the SIG models are recommended, but among the gamma models the RE model is considered as the best fitting model.

The Q-Q plots for this case study are constructed by considering the method described in the GaAs laser data analysis developed in Section 6.1. In Fig. 6, the Q-Q plots for the different gamma models are presented, it can be noted that the gamma RE model has the best fitting for the data. The obtained AD values were 1.860, 0.620, 1.863, 1.864 for the SGP, RE, RV, and RM models, respectively. By also considering the normal critical value of the AD statistic for a 0.1 significance level of 0.63, it can be noted that the RE model is not rejected. For this particular case study, it can be noted that the variation in the degradation rate and the variation of the degradation increments within each unit in the degradation paths in Fig. 4 are significant. Since the random effects of the RE model are involved in the degradation mean and variance, it is expected that the degradation paths vary significantly. Thus, this model is suitable for degradation modeling of products where significant product heterogeneity is observed, such as the case study in this section.

7. Concluding remarks and discussion

It is common that the degradation rate for products that have been manufactured by stable and mature processes tends to have a low level of variation. In addition, a short variation within the degradation paths may be expected. However, for processes in a tenure phase of development, several unknown factors that cause variation in the degradation paths are expected to play an important role in the reliability assessment of the products. Identifying the effect of such unknown factors over the degradation process of the product under study can lead to determine if both the rate and the volatility of the process are affected. But, it is important to consider that for some products, just the rate or the volatility of the process are affected by the random effects. When this is the case, the classical gamma process with random effects fails to model such scenarios. Considering the scale parameter of the gamma process as random, it is possible to note that both the mean and variance of the degradation process are affected. A simple modification of the structure of the parameters of the gamma process is proposed in this study in order to obtain random effects models that solely affect the degradation mean or the variance of the degradation process. In such models, it is possible to consider cases when a high degradation rate and a low level of variation within the degradation paths are observed for the random mean model and cases when a low degradation rate and a large variation within the degradation paths are observed for the random variance model. The proposed models were implemented along with the simple gamma process and the classical gamma process with random effects in two case studies. It was found that in the GaAs laser degradation dataset, the RM model fits better the data, in such case study it is possible to note a low level of variation of the within increments of the degradation paths and a large variation of the degradation rate, which explains the best fitting of the RM model. In the second case study, the degradation paths are characterized by a large variation in both the degradation rate and the variance within the degradation paths, in such case study it was found that the best fitting model is the classical gamma process with random effects. Which denotes that the proposed models are supposed to fit specific cases of random effects models, as previously described. The proposed gamma random effects models were compared to the IG models with random effects by fitting the degradation datasets to the IG models. For the GaAs laser data, it was found that the gamma RM model is superior to all of the IG models. On the other hand, for the fatigue-crack growth data it was found that the IG models are favored over all the gamma models. In such example, the SIG mode is favored by the information criterion. Future research can be directed to consider the shape parameter of the gamma process as a random effect parameter, in such case the both the mean and variance of the process are affected. However, a modification of the structure of the parameters can lead to specific random effects models such as the ones described in this paper.

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