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Aiding decision makers in articulating a preference closeness model through compensatory fuzzy logic for many-objective optimization problems

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A R T I C L E I N F O

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A B S T R A C T

One of the main challenges in applying preference-based approaches to many-objective optimization problems is that decision makers (DMs) initially have only a vague notion of the solution they want and can obtain. In this paper, we propose an interactive approach that aids DMs in articulating a preference model in a progressive way. The quality of a solution is determined in terms of its "preference closeness" to an aspiration point, which is a subjective concept that can be outlined by the DM. Our proposal is based on compensatory fuzzy logic, which allows for the construction of predicates that are expressed in language that is close to natural. One main advantage is that the model can be optimized via metaheuristics, and we utilize an ant colony optimization algorithm for this. Our model complies with the principles of hybrid augmented intelligence, not only because the algorithm is enriched with knowledge from the DM, but also because the DM also learns the concept of "preference closeness" throughout the process. The proposed model is validated on benchmarks with five and 10 objective functions, and is compared with two state-of-the-art algorithms. Our approach allows for better convergence to the best compromise solutions. The advantages of our approach are supported by statistical tests of the results.

1. Introduction

Multi-objective evolutionary algorithms (MOEAs) have become a leading way of addressing the challenges that arise from complex multi-objective optimization problems (MOPs). MOEAs have been shown to be able to find an approximation to the Pareto frontier for many problems. In this case, the decision maker (DM) must identify the best compromise solution, i.e., the one that is most in agreement with their particular preferences and values, within a privileged preference zone (region of interest, RoI) [1]. However, as the number of objectives increases, the search space becomes highly dimensional, making it challenging to find true Pareto solutions and to identify the most preferred solution. MOPs are complex due to the conflicting nature of the objectives and the multidimensionality of the search space.

MOPs with more than three objectives are known in the literature as many-objective optimization problems (MaOPs). To enable them to be applied to such problems, several essential challenges associated with MOEAs must be overcome, such as an exponential increment of non-dominated solutions and a degradation of the selective pressure towards the Pareto frontier caused by the increase in the dimensionality of the objective space [2,3]. This high dimensionality makes visualization challenging [4] and finding a suitable set of Pareto optimal solutions difficult [5]. These and other aspects are discussed in the literature [6,7].

Currently, algorithms with a focus on solving MaOPs are being developed and tested (e.g., [2,8]). The aim of these algorithms is to find

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a convergent, representative, well-distributed Pareto frontier sample. However, more work is needed to solve MaOPs completely. It should be highlighted that the final solution is always related to a DM's preferences, and these preferences must, therefore, be incorporated in some way in order to identify the "best" solution.

Three key aspects must be considered when designing a method for incorporating preferences into MaOPs:

1. The stage or phase of the search in which the preferences are incorporated (i.e., *a priori*, progressively, or *a posteriori*);
2. The preferential information that must be provided by the DM;
3. The multi-criteria preference model to be used.

Regarding the first point, the progressive (interactive) incorporation of preferences offers certain advantages (e.g., [9–13]):

- It is accepted that interactive approaches help DMs "learn" about their problem, mainly with regard to the complex interactions among the objective functions and the trade-offs they will need to make on the Pareto front. The DM's preferences are updated at each interaction step, enabling better solutions to be identified.
- DMs tend to feel more comfortable with the results of interactive procedures, as they are involved in the search process and have systematically accepted its output as satisfactory solutions.
- Evidence shows that interactive approaches reduce the search space and help in finding the RoI [14,15], particularly for MaOPs [16–18]. In view of their capacity to offer ever more focused lists of solutions to DMs, these approaches have become known as "pruning methods" based on the DM's preferences and knowledge [19].

Some of the disadvantages of interactive preference incorporation methods are as follows:

- Difficulty in making preference judgments in regard to solutions to problems with many objectives. To avoid irrational behaviors, interactive methods often require transitive judgments from DMs [20]; these types of demand contradict, to a great extent, the cognitive limitations of the human mind (cf. [21]) and may be hard to fulfill when the DM needs to compare, rank, or judge solutions described by five or more conflicting objectives.
- Difficulty in establishing friendly and fluid communication between DMs and optimization algorithms. This is mainly because DMs are often not familiar with the formal and technical aspects of the preference models implemented in the method.

Aspects 2 and 3 are closely related. The use of reference points or aspiration levels (including the ideal point as a particular case) is perhaps the most popular way to incorporate preferences into MOEAs, as many DMs feel comfortable expressing their preferences in terms of aspiration levels. Examples of this strategy can be found in studies by Aggarwal and Mishra [22], He et al. [23], Jiang et al. [24], Li et al. [25], Ngo et al. [26], Peng et al. [27], Yu et al. [28], Zhao et al. [29], and Zou et al. [30,31] Indeed, reference points or aspiration levels have been used over time as part of various strategies for solving problems with multiple criteria or objectives, for example:

(i) TOPSIS [32] and VIKOR [33] in the context of multi-criteria decision analysis; and
(ii) conventional multi-objective optimization techniques such as goal programming (e.g., [34]), compromise programming (e.g., [35]), and scalarizing functions (e.g., [36]).

When aspiration levels are used to represent preferences, it is assumed that the propositions "x is closer than y to the aspiration point" and "x is preferred to y" are equivalent. This assumption becomes critical in regard to the selection of an appropriate closeness measure. Many different metrics are used in methods based on aspiration levels to assess the nearness of solutions to an aspiration point, and these metrics are not free of arbitrariness. An appropriate metric should measure the extent to which a solution is satisfactory to the DM, and should model the "preference closeness" to their aspiration point. Indeed, the best compromise solution can be defined as the Pareto optimal solution that maximizes the preference closeness to the aspiration point.

Since both "closeness" and "preference closeness" are vague concepts, we propose using fuzzy logic to model them. Fuzzy logic is of the utmost importance in many areas due to its ability to handle imprecision, vagueness, and ambiguity, which are features of real-world problems (e.g., [25,37–40,]). Fuzzy logic is a powerful tool for building a mathematical model using vague statements expressed in natural language [41]. This ability is essential in creating a friendly "preference closeness" model, as it facilitates the interaction between DMs and interactive optimization algorithms. Although fuzzy sets have been extensively used in conventional multi-objective optimization [42], to the best of our knowledge, this approach has not previously been proposed as a tool for modeling the preference closeness to the best compromise in MOEAs and MaOEAs. Hence, it is not an option in the different available surveys related to incorporating preferences.

This paper presents a fuzzy preference model that is integrated into an interactive method for learning and updating preferences, thereby creating a hybrid-augmented intelligence (HAI) system for addressing MaOPs. Our approach is aligned with the original concept developed by Zheng et al. [43], wherein an HAI system introduces human interaction into an artificial intelligence (AI) model. As time progresses, the AI model enriches (augments) human expertise, knowledge, and experience [18]. The implementation of effective interaction represents an important challenge in terms of handling learning preferences and ensuring satisfactory results for complex problems [44] such as MaOPs.

This paper makes several contributions to the current literature:

− We present a preference closeness model of the best compromise, based on an aggregation of the component predicates regarding the nearness of each objective function to an aspiration point and its importance within the set of objectives (for more flexibility, the information on the importance of each objective should be purely ordinal).
− The DM's multi-criteria preferences are represented using a fuzzy predicate based on plain natural language statements regarding the closeness to their aspiration point.
− We propose an interactive method in which DMs—taking into account the truth values of a predicate regarding the preference closeness combined with their judgments about the current solutions—can clarify their actual preferences, modify ill-shaped preferences and values in their mind, and update the logical model of preference closeness to the best compromise. This interactive method is robust with regard to the increment in the number of objective functions and possible intransitive judgments.

To validate the contributions of our work, we carry out the following:

− We perform simulations of the DM's learning process and updated preferences through their interaction with the search algorithm;
− We create a powerful enriched metaheuristic that is able to exploit the model of preference closeness; and
− We test this enriched metaheuristic on a wide range of challenging benchmark problems with many objective functions, including a comparison with some state-of-the-art metaheuristics.

The paper is structured as follows: some concepts related to preference closeness and compensatory fuzzy logic are briefly reviewed in Section 2, and in this context, the fuzzy preference model and the interactive learning process are discussed in Section 3. Section 4 outlines the ant colony optimization (ACO) algorithm used to address the fuzzy predicate about preference closeness. Some computer experiments conducted to demonstrate the potential of the new approach, its robustness with respect to increasing dimensionality, and its advantages in comparison with other state-of-the-art algorithms are described in Section 5. Lastly, our conclusions and some directions for future research are discussed in Section 6.

2. Background

This section introduces some background required to understand our contributions adequately. Section 2.1 gives an overview of the origin and evolution of the concept of "preference closeness", and Section 2.2 reviews the basis of compensatory fuzzy logic.

2.1. Concept of preference closeness

In computational intelligence, preference closeness describes the degree of proximity—from the point of view of a preference—between the outcomes of a multi-criteria decision problem or recommendation system. A high degree of preference-based proximity is equivalent to a statement of indifference towards the outcomes.

The idea of maximizing a preference closeness to the ideal point may have originated with TOPSIS [32], a method developed for multi-criteria decision-making. Its primary idea relies on identifying solutions with the shortest preference-based distance from the ideal solution and the farthest preferential distance from the negative-ideal solution.

Several interpretations of preference closeness can be found in the scientific literature. Fig. 1 presents a timeline of the main contributions to the notion of preference closeness. The underlying principle of maximizing preference closeness has also been used in various applications, including technology transfer efficiency, industrial robotic systems evaluation, and wastewater treatment technology evaluation [17,45].

The application of preference closeness to MaOPs represents a research gap, which this study aims to narrow by using a model based on compensatory fuzzy logic to articulate the DM's preferences.

2.2. Compensatory fuzzy logic

Preferential knowledge is the understanding that a DM can generally provide, which involves the explicit articulation of preferences and decision-making heuristics in their field of expertise to reach a final decision [46]. There are several alternative approaches in the literature with regard to the challenge of building preferential knowledge. Although AI typically focuses on diagnosis and knowledge representation, it often does not model human preferences, and limits subjectivity. Decision analysis methods, including functional and relational approaches, lead the modeling of the DM's multi-criteria preferences and risk attitudes, and dismiss human experience and reasoning [20,50].

Learning from examples using AI has become an important approach to decision-making. Dominance-based rough set methodology is probably the best representative of an approach that bridges AI and multi-criteria decision analysis [51]. However, learning from examples lacks

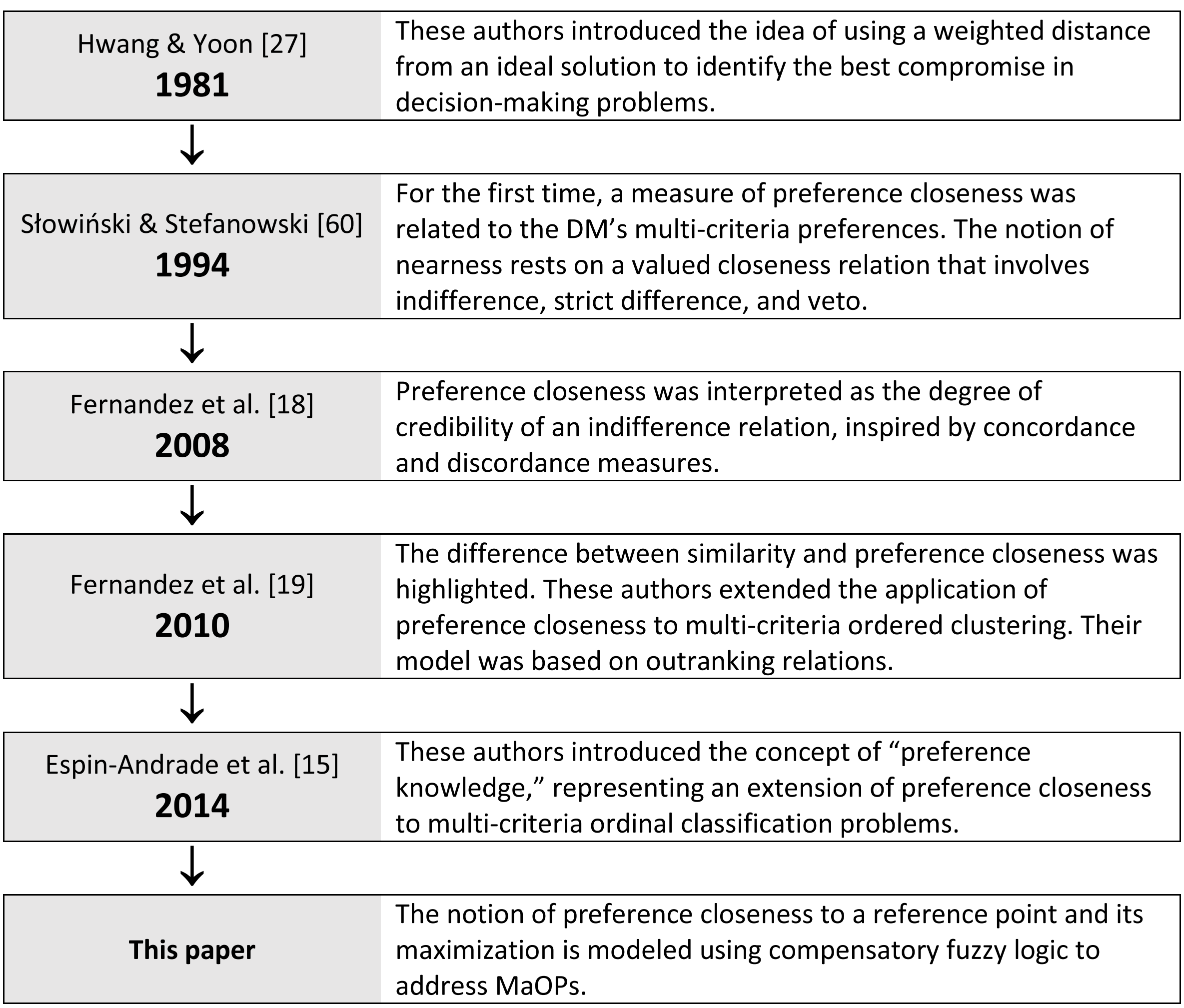


Fig. 1. Timeline of contributions related to preference closeness [32,46–49].

an explicit model of preferences, and limitations arise in terms of handling conflicting attributes as the number of attributes increases, due to the cognitive limitations of humans.

Building preferential knowledge, which emphasizes the explicit representation of a DM's preferences and reasoning, remains an open area of research. Today, the main challenge lies in using AI technologies to represent this preferential knowledge, including the ways in which DMs aggregate multi-criteria information and reasoning to arrive at a final decision [52].

Fuzzy logic has enabled advances in using verbal expressions to describe preferences and decision-making heuristics. It has been used to model preference representation and reasoning in a flexible and refined way; for example, rather than using strict binary choices such as "like" and "dislike," fuzzy preference models can capture degrees of preference [53,54].

Compensatory fuzzy logic is a recent axiom-based approach that is compatible with the preferential reasoning that characterizes realistic decision-making processes [55]. This approach extends traditional fuzzy logic by exploiting its benefits in terms of modeling decision-making scenarios where there are compensatory effects between multiple criteria. It allows for a trade-off between different criteria, where a deficiency in one criterion can be compensated to a certain extent by excellence in another. In general, compensatory fuzzy logic removes classical axioms to give a sensitive and idempotent system. Unlike conventional fuzzy logic, where the truth value of a conjunction is always smaller than or equal to the truth values of its components, and the truth value of a disjunction is always greater or equal, compensatory fuzzy logic removes these constraints. This foundational change enables compensatory fuzzy logic to compensate for decreased truth values in one component predicate by increasing another, thereby allowing for higher conjunction values.

An instance of compensatory fuzzy logic is defined as a quartet of continuous operators ($\wedge$, $\vee$, o, $\neg$), (representing conjunction, disjunction, strict order, and negation, respectively), which satisfy a set of axioms stated by Espin-Andrade et al. [46]. Some of these axioms are the following:

- Compensation: $\min\{x_1, x_2, \ldots, x_n\} \le \wedge(x_1, x_2, \ldots, x_n) \le \max\{x_1, x_2, \ldots, x_n\}$;
- Strict growth: if $x_1 = y_1,\ x_2 = y_2,\ \ldots,\ x_{i-1} = y_{i-1},\ x_{i+1} = y_{i+1},\ \ldots,\ x_n = y_n$ are different from zero and $x_i > y_i$, then $\wedge(x_1, x_2, \ldots, x_n) > \wedge(y_1, y_2, \ldots, y_n)$;
- Veto: If $x_i = 0$ for a given i, then $\wedge(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) = 0$;
- Symmetry: $\wedge(x_1, x_2, \ldots, x_i, x_j, \ldots, x_n) = \wedge(x_1, x_2, \ldots, x_j, x_i, \ldots, x_n)$;
- Fuzzy transitivity: if $o(x, y) \ge 0.5$ and $o(y, z) \ge 0.5$, then $o(x, z) \ge \max\{o(x, y), o(y, z)\}$.

Espin-Andrade et al. [46] argued that the axioms given above combine rational features of the functional approach to decision-making with the veto capacity, a characteristic of some outranking approaches.

Each instance of compensatory fuzzy logic is defined by several fundamental operators. Geometric mean-based compensatory fuzzy logic (GMCFL) was proposed by Espin-Andrade et al. [46] as the first instance of a compensatory fuzzy logic. The geometric mean is a kind of quasi-arithmetic mean that forms the class of compensatory operators that fulfill the set of axioms of compensatory fuzzy logic. Like any compensatory operator, the geometric mean satisfies the property of idempotence; that is, $\wedge(x, x, \ldots, x) = x$. This is one of the most widely used operators of the quasi-arithmetic mean class. The geometric mean is an instance of the ordered weighted geometric operator, which has desirable properties for fuzzy-based decision-making [56].

In addition, the geometric mean is simpler than most quasi-arithmetic means. Several fundamental operators and properties of the GMCFL are shown in Table 1. It should be noted that the disjunction operator in Table 1 fulfills the same properties of the geometric mean.

Table 1
Some operators of the GMFCL.

1. Conjunction $\wedge(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} (x_i)\right)^{1/n}$	2. Disjunction $\vee(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - \left(\prod_{i=1}^{n} (1 - \boldsymbol{x}_i)\right)^{1/n}$
3. Negation $\neg(\boldsymbol{x}) = 1 - \boldsymbol{x}$	4. Strict order $o(x, y) = 0.5[x - y] + 0.5$

Espin-Andrade et al. [55] proved that GMCFL is compatible with Archimedean logic based on the popularly used product t-norm as the conjunction operator. Compatibility means that any order of the predicates in the universe is the same when both logic systems are applied. Hence, the maximum value of truth of a predicate defined on a set corresponds to the same element of the set, and it does not matter which compatible logic system is used to assess truthfulness. Indeed, the unique compensatory logic that fulfills this feature regarding the product t-norm is the GMCFL.

3. Learning the notion of preference closeness to a reference point

We propose a preference model based on reference points, and assume that the DM can judge the solutions based on their closeness to such points. Here, we differentiate between two reference points: aspiration and reservation. The aspiration point is an m-dimensional vector with the values the DM would like to achieve; in contrast, the reservation point contains the values the DM wants to avoid.

Our model works based on the following fundamental premises:

Assumption 1. The best compromise solution is a point on the Pareto front where the preference closeness to the aspiration point is a maximum.

Assumption 2. The DM can progressively improve the preference model by judging the solutions generated during the search process.

Assumption 3. Using natural language statements, the DM can express their current understanding of preference closeness to their aspiration point in terms of nearness to the aspiration point and the priority attached to each objective.

Regarding Assumption 1, we note that all models based on aspiration points rest on a similar premise. We also note that Assumption 2 is a core premise of interactive approaches. Assumptions 1 and 2 then conjointly demand the same conditions of any interactive approach based on reference points (cf. [25]). Consequently, Assumption 3 is the only unique condition required by our model. Obviously, no preference model is free of all assumptions about the DM's capabilities or behavior, and each of them assumes the capacity of the model to match the actual preferences of a subset of DMs. A typical DM is likely to feel more comfortable expressing their preferences using natural language-like predicates than initializing complex mathematical models. This advantage is connected to our model based on fuzzy logic. As an additional benefit, it is interpretable, allowing DMs to provide clear reasons that justify their final decision.

The model proposed in this paper includes several optimization phases. After each phase, the solutions are presented to the DM for perusal. This approach allows the DM to "learn" an appropriate notion of preference closeness and to progressively update the aspiration levels. After each interaction, the DM gathers knowledge about the optimized solutions, mainly in terms of the range of the objective functions and their trade-offs. This knowledge can be expressed through compensatory fuzzy logic. Based on Assumptions 1–3, the DM's learning process is illustrated in Fig. 2. This should be considered as only one possible learning path, since other ways exist to express the current notion of preference closeness that are compatible with the above assumptions. In the following, we describe each step of our approach.

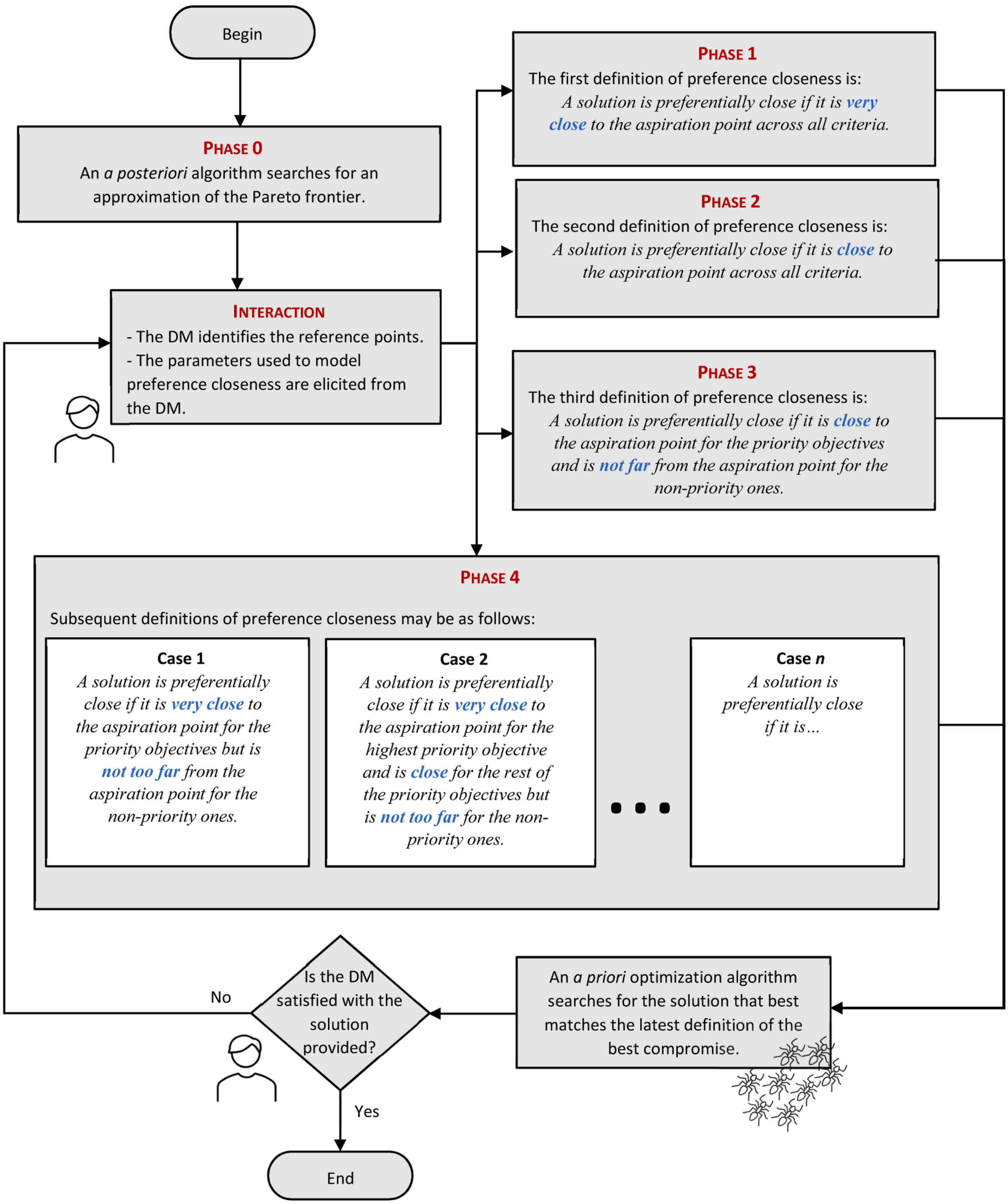


Fig. 2. Flowchart for the process of preference articulation.

- Phase 0

At this stage, the objectives are to explore the search space and to obtain an approximate sample of the Pareto frontier.

- Interaction

First, the solutions are presented to the DM, who then proposes the reference points. Without loss of generality, this model focuses on minimizing objectives. Let $\mathcal{O}$ be a set of actions assessed based on m criteria (known as "objective functions" in the jargon of optimization). We can consider the following axioms:

(i) The most stringent instance of the aspiration point is the ideal point; in other words

$$\mathscr{A} = \{z_1^{\mathrm{id}},\ z_2^{\mathrm{id}}, \ldots, z_i^{\mathrm{id}}, z_{i+1}^{\mathrm{id}}, \ldots, z_m^{\mathrm{id}}\}, \tag{1}$$

where $\mathscr{A}$ stands for the aspiration point and z_i^{id} is defined as

$$z_i^{\mathrm{id}} = \min_{x \in PS(\mathscr{O})} \{z_i(x)\}\ \forall i \in \{1,\ 2,\ 3, \ldots, m\}, \tag{2}$$

where $z_i(x)$ is the value of x in the ith objective function, $\mathscr{O}$ is the solution set, and $PS(\mathscr{O})$ is the subset of Pareto-efficient solutions.

(ii) The loosest instance of the reservation point is the nadir point of $\mathscr{O}$; in other words

$$\mathscr{R} = \{z_1^{\mathrm{nad}},\ z_2^{\mathrm{nad}}, \ldots, z_i^{\mathrm{nad}}, z_{i+1}^{\mathrm{nad}}, \ldots, z_m^{\mathrm{nad}}\}, \tag{3}$$

where $\mathscr{R}$ stands for the reservation point, and z_i^{nad} is defined as

$$z_i^{\mathrm{nad}} = \max_{x \in PS(\mathscr{O})} \{z_i(x)\}\ \forall i \in \{1,\ 2,\ 3, \ldots, m\}. \tag{4}$$

Derived from Assumption 1, the feasible region for aspiration and reservation points is bounded by the ideal and nadir points of $PS(\mathscr{O})$. Consequently, it is impossible to reach solutions with values better than the ideal point, as stated in (i). In addition, the DM should not settle for values that are worse than the nadir point, to ensure that the best compromise is Pareto-efficient, as stated in (ii). For the sake of simplicity, the aspiration and reservation points are instantiated here as the ideal and nadir points of the Pareto-efficient solutions known so far. Nevertheless, our model is compatible with any other instance fulfilling (i) and (ii) and $\mathscr{A}_i < \mathscr{R}_i\ \forall i \in \{1,\ 2,\ 3,\ \ldots, m\}$ (i.e., the aspiration point is strictly better than the reservation point for every single criterion).

Once the reference points are known, their distances can be computed. Let $\delta_i(x)$ represent the proportional difference between z_i^{id} and $z_i(x)$ at the ith coordinate:

$$\delta_i(x) = \frac{z_i(x) - z_i^{\mathrm{id}}}{z_i^{\mathrm{nad}} - z_i^{\mathrm{id}}}\ \forall i \in \{1,\ 2,\ 3, \ldots, m\}. \tag{5}$$

We can then model how the DM perceives such distances. Consider the fuzzy linguistic variable

a_i : x is close to the aspiration point at the ith coordinatewith the following membership function

$$\mu(a_i) = 1 - s(\delta_i(x), \alpha,\ \gamma), \tag{6}$$

where $s(\delta_i(x), \alpha,\ \gamma)$ is a decreasing sigmoid function with parameters α and γ, defined as

$$s(\delta_i(x), \alpha,\ \gamma) = \frac{1}{1 + e^{-\alpha(\delta_i(x) - \gamma)}}. \tag{7}$$

Note that α and γ should be set to reflect the DM's preference in regard to the ith criterion. One advantage is that this task can be performed during the interaction. γ represents the point where $\mu(a_i) = 0.5$, which can be directly elicited after presenting the solutions to the DM and asking for a point where the value is as true as it is false. α is the growth rate, which can be isolated from Eq. (7) by asking for a point where the DM is sure that $\mu(a_i) \approx 1$. Some examples of eliciting α and γ are presented by Espin-Andrade et al. [41]. Fig. 3 depicts an example of $\mu(a_i)$.

Using the sigmoid function as a membership function has several beneficial properties, especially when it involves optimization. It mainly has asymptotes, thus ensuring a greater degree of truth if further optimized values are found.

We can now model the linguistic variable

ϑ_i : x is very close to the aspiration point at the ith coordinatewith the fuzzy membership function

$$\mu(\vartheta_i) = [\mu(a_i)]^2. \tag{8}$$

- Phase 1

Initially, the DM would like to obtain a solution that is as preferentially close as possible to their aspiration point, which corresponds to the predicate

$$\mathscr{P}_1(z(x)) = \bigwedge_{m}^{i=1} \mu(\vartheta_i) \tag{9}$$

Eq. (9) represents the degree of truth for the statement "x is very close to the aspiration point across all criteria." Suppose there are solutions with acceptable values (typically greater than 0.85). In this case, the best compromise is the solution with the highest degree of truth, which can be modeled as

$$\mathscr{P}_1^{\star} = \arg\max_{x \in PS(\mathscr{O})} \{\mathscr{P}_1(z(x))\}. \tag{10}$$

$\mathscr{P}_1^{\star}$ is the best compromise for Phase 1 of this preference model, and an *a priori* optimization algorithm can be used to approximate it. Since objective functions often conflict in MaOPs, the DM may not be completely satisfied with the provided solution. The model then offers a second phase with a lower level of strictness.

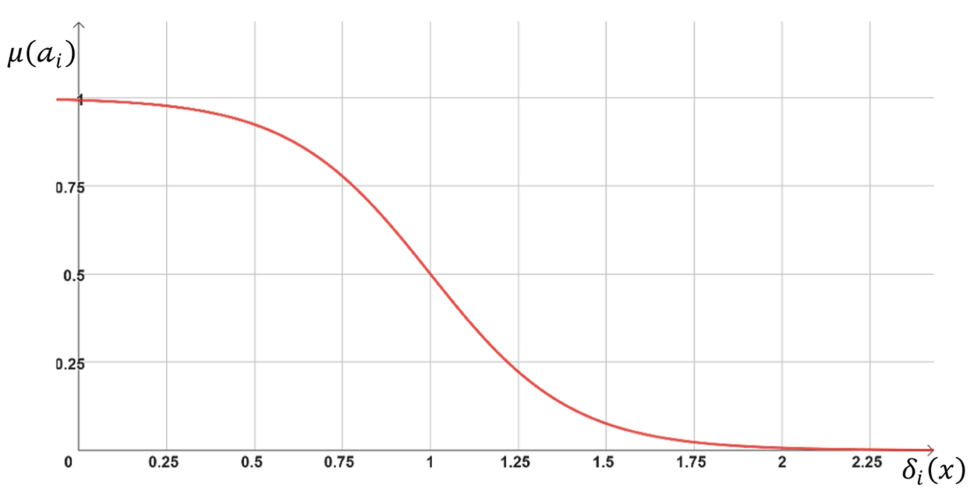


Fig. 3. Plot of $\mu(a_i)$ for values of $\gamma = 1$ and $\alpha = 5$. The x-axis represents the proportional distance from a solution x to the aspiration point at the ith coordinate, and the y-axis represents the truth degree of the linguistic variable "x is close to the aspiration point at the ith coordinate."

- Phase 2

Consider the following fuzzy predicate

$$\mathscr{P}_2(z(x)) = \overset{i=1}{\underset{m}{\wedge}} \mu(\alpha_i) \tag{11}$$

representing the degree of truth for the statement "x is close to the aspiration point across all objectives." Then, the best compromise during Phase 2 should be

$$\mathscr{P}_2^{\star} = \underset{x\in PS(\mathscr{O})}{\arg\max}\{\mathscr{P}_2(z(x))\}, \tag{12}$$

which would match the DM's preference if it has a high value.

Suppose there is no solution satisfying the DM's requirements, which implies that no solution is preferentially close to their aspiration point across all objective functions simultaneously. The best compromise should then be reformulated in terms of not only proximity but also farness from the aspiration point.

- Phase 3

We can model the fuzzy linguistic variable

$\mathscr{e}_i$: x is far from the aspiration point at the ith coordinatewith the following membership function

$$\mu(\mathscr{e}_i) = s'(\delta_i(x), \alpha', \gamma') = \frac{1}{1 + e^{-\alpha'(\delta_i(x) - \gamma')}}. \tag{13}$$

Note that $s'(\delta_i(x), \alpha', \gamma')$ is an increasing sigmoid function with parameters α' and γ', which should be elicited during an interaction with the DM. It is possible that the DM would like a solution that is close to their aspiration point for certain objectives they consider more important, and may be prepared to tolerate losses in the less important ones. Let $\mathbb{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \ldots\}$ be the set of priority objectives, where $1 \le \mathbb{P}_i \le m$.

Consider the following fuzzy predicate

$$\mathscr{P}_3(z(x)) = \overset{i=1}{\underset{m}{\wedge}} g_i \tag{14}$$

where

$$\mathscr{g}_i = \begin{cases} \mu(\alpha_i) & \text{if } i \in \mathbb{P}, \\ \neg\mu(\mathscr{e}_i) & \text{otherwise.} \end{cases} \tag{15}$$

$\mathscr{P}_3(z(x))$ is the degree of truth for "x is close to the aspiration point in the priority objectives but is not far in the non-priority ones." Accordingly, the best solution in Phase 3 is

$$\mathscr{P}_3^{\star} = \underset{x\in PS(\mathscr{O})}{\arg\max}\{\mathscr{P}_3(z(x))\}. \tag{16}$$

If $\mathscr{P}_3^{\star}$ does not satisfy the DM, Phase 4 is executed.

- Phase 4

Here, the DM may propose any predicates involving the priority of the objectives and distances to their aspiration point. The DM may vary the distance modifiers and priority levels. We study the following two cases:

Case 1. *A solution is preferentially close if it is very close to the aspiration point for the priority objectives and is not too far from the aspiration point for the non-priority ones.*

The intensifier "too much" can be applied to Eq. (13) as follows:

$$\mu(\mathscr{d}_i) = [\mu(\mathscr{e}_i)]^2. \tag{17}$$

Hence, the degree of truth for the statement "x is very close to the aspiration point for the priority objectives and is not too far for the non-priority ones" should be considered as

$$\mathscr{P}_4(z(x)) = \overset{i=1}{\underset{m}{\wedge}} h_i \tag{18}$$

where

$$\mathscr{h}_i = \begin{cases} \mu(\mathscr{b}_i) & \text{if } i \in \mathbb{P}, \\ \neg\mu(\mathscr{d}_i) & \text{otherwise.} \end{cases} \tag{19}$$

Case 2. *A solution is preferentially close if it is very close to the aspiration point for the highest priority objective and is close for the rest of the priority objectives, and is not too far for the non-priority ones.*

The degree of truth for this predicate should be modeled as

$$\mathscr{P}_4(z(x)) = \overset{i=1}{\underset{m}{\wedge}} f_i \tag{20}$$

where

$$\mathscr{f}_i = \begin{cases} \mu(\mathscr{b}_i) & \text{if i} = \mathbb{P}^*, \\ \mu(\alpha_i) & \text{if } i \in \mathbb{P}, \\ \neg\mu(\mathscr{d}_i) & \text{otherwise,} \end{cases} \tag{21}$$

where $\mathbb{P}^*$ is the highest priority objective. Regardless of the case, the best solution in Phase 4 is

$$\mathscr{P}_4^{\star} = \underset{x\in PS(\mathscr{O})}{\arg\max}\{\mathscr{P}_4(z(x))\}. \tag{22}$$

If the degree of truth of $\mathscr{P}_4^{\star}$ is still low, the DM should:

(1) Accept $\mathscr{P}_4^{\star}$ as the best compromise even though they are not fully satisfied. The solution that the DM wants cannot be generated by the search algorithm (and may even be unfeasible).
(2) Try some of the following measures:
(a) reducing the number of priority objectives,
(b) relaxing their notion of preference closeness by changing the parameters of the sigmoid function (α and γ),
(c) attaching different priority levels to the objectives (e.g., *highest, high, medium*, and *low*), and trying a combination of different linguistic states (e.g., *very close, close, far*, and *too far*). The DM should then proceed with the optimization algorithm using the new predicate.

We would like to highlight the versatility of our framework. In problems complying with the three fundamental premises presented at the beginning of this section, DMs will discover their own definition of preference closeness by composing different prompts, modeled as fuzzy predicates, that serve as the preference model they want to optimize.

4. Optimization algorithm

In this study, we extend $\text{ACO}_{\mathbb{R}}$—ant colony optimization for continuous domains [57]—to address the preference model. The rationale for this choice is given as follows:

- $\text{ACO}_{\mathbb{R}}$ is the most representative ACO algorithm for optimization problems with continuous decision variables (cf. [58]).
- It has been successfully applied to MaOPs, with acceptable convergence to the Pareto frontier (e.g., [59]).
- $\text{ACO}_{\mathbb{R}}$ has been extended to incorporate the DM's preferences in many-objective optimization, with competitive performance (cf. [58]).

Table 2
Structure of the pheromone matrix in our ACO algorithm.

$\tau_1 =$	$\tau_{1,1}$	$\tau_{1,2}$	$\cdots$	$\tau_{1,i}$	$\cdots$	$\tau_{1,n}$	$f(\tau_1)$	ω_1
$\tau_2 =$	$\tau_{2,1}$	$\tau_{2,2}$	$\cdots$	$\tau_{2,i}$	$\cdots$	$\tau_{2,n}$	$f(\tau_2)$	ω_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\tau_\kappa =$	$\tau_{\kappa,1}$	$\tau_{\kappa,2}$	$\cdots$	$\tau_{\kappa,i}$	$\cdots$	$\tau_{\kappa,n}$	$f(\tau_\kappa)$	ω_κ

- The pheromone matrix can support progressive models, and provides encouraging results when several interactions with the DM are performed during optimization (e.g., [60]).

Considering the above remarks, an extension of $\text{ACO}_{\mathbb{R}}$ seems adequate for embedding the proposed model. We modify the pheromone matrix, taking as our objective function the degree of truth of the predicate used to assess the DM's preference. Note that the objective function varies depending the phase being optimized. A particular case is Phase 0, which uses the R2 score [61]. R2 is an indicator of the uniform distribution of vector solutions. Consequently, the ACO algorithm behaves like an *a posteriori* approach (in other words, it searches for a well-distributed sample of the complete Pareto frontier) before the DM sets any preferences for the solutions. Table 2 depicts the pheromone matrix used in this paper.

In Table 2, κ is the size of the colony, n is the number of decision variables, and $\tau_j = \langle \tau_{j,1},\ \tau_{j,2},\ \tau_{j,3}, \ldots,\ \tau_{j,n} \rangle$ is the jth solution, with values $\tau_{j,i}\ \forall i \in \{1, 2, 3, \ldots,\ n\}$ in the decision variables. $f(\tau_\kappa)$ is the value of the preference model, specifically:

$$f(\tau_\kappa) = \begin{cases} R2(z(\tau_\kappa)) & \text{before the first interaction,} \\ \mathscr{P}_1(z(\tau_\kappa)) & \text{after the first interaction,} \\ \mathscr{P}_2(z(\tau_\kappa)) & \text{after the second interaction,} \\ \mathscr{P}_3(z(\tau_\kappa)) & \text{after the third interaction,} \\ \mathscr{P}_4(z(\tau_\kappa)) & \text{after the fourth interaction.} \end{cases} \tag{23}$$

Note in Table 2 that the solutions are kept sorted by $f(\tau_\kappa)$ in decreasing order of preference. The last column is ω_j, defined as:

$$\omega_j = \frac{e^{-\varphi(j)}}{\varsigma \cdot \kappa \sqrt{2\pi}},\ \text{where}\ \varphi(j) = \frac{(j-1)}{2\varsigma^2\kappa^2}. \tag{24}$$

In Eq. (24), ω_j acts like the weight of τ_j, which depends on its position. The weights are calculated through a normal function (argument j, mean 1.0) with standard deviation $\varsigma \cdot \kappa$. Here, ς is the parameter that sets the exploitation-exploration balance ($0 < \varsigma \leq 1$).

The colony constructs new solutions by taking the rows of the pheromone matrix as variable distributions, which it samples. A new solution $x_j = \langle x_{j,1},\ x_{j,2}, x_{j,3},\ \ldots, x_{j,n} \rangle$ takes values following:

$$x_{j,i} \sim g_\imath^i(x)\ \forall i \in \{1,\ 2,\ 3,\ \ldots,\ n\}\ . \tag{25}$$

In Eq. (25), $\imath$ is a row of the pheromone matrix, which is selected through a roulette wheel function on the weights, and $g_\imath^i(x)$ is a normal function, defined as

$$g_\imath^i(x) = \frac{e^{-\phi_i(\imath)}}{s_\imath^i\sqrt{2\pi}},\ \text{where}\ \phi_i(\imath) = \frac{(x - x_{\imath,i})^2}{2(s_\imath^i)},\tag{26}$$

where $s_\imath^i$ is the standard deviation, which is calculated as the colony sample solutions over the iterations as follows:

$$s_\imath^i = \xi \sum_{J=1}^{\kappa} \frac{|x_{J,i} - x_{\imath,i}|}{\kappa - 1} \tag{27}$$

Algorithm 1 outlines the ACO extension used for optimization. The time-complexity function of Algorithm 1 is $O(\kappa(\kappa m + n))$, where κ is the size of the colony, m is the number of objectives, and n is the number of decision variables (cf. [62]). We encourage the reader to consult Socha and Dorigo [57] for a more detailed description of the optimization algorithm.

Algorithm 1
Outline of the ACO metaheuristic.

```
Initialize: 𝒜←∅, iter←0
While iter < iter_max
    For each ant in the colony
        A←buildNewSolution() // Using Eq. (25)
    End for
    τ←τ ∪ A
    Sort τ // See Table 2
    Remove from τ the last solutions exceeding κ
    If an interaction is required
     updatePreferenceModel(); // Section 3
     End if
    iter←iter + 1
End while
```

5. Results

This section presents some numerical results to illustrate the benefits of our approach. In Section 5.1, we describe the setting for the experiments. Section 5.2 gives an illustrative application of our approach. Lastly, Section 5.3 compares the results with two state-of-the-art algorithms, the reference-vector-guided evolutionary algorithm with improved growing neural gas, shortened to RVEA-iGNG [63], and the two-stage non-dominated sorting genetic algorithm II, shortened to TS-NSGA-II [64], on two classic test suites:

- The Deb, Thiele, Laumanns, and Zitzler problems [65], abbreviated as DTLZ.
- The Walking Fish Group problems [66], abbreviated as WFG.

5.1. Experimental conditions

We coded the ACO algorithm using standard C in Linux (Ubuntu 18) on a computer with an Intel Core i7-6700 at 3.4 GHz and 16GB of RAM.

The ACO's parameters were set to $\varsigma = 0.1$ and $\xi = 0.05$. These values were identified by experimentation, and represent the best combination from $(\varsigma, \xi) \in \{0.01,\ 0.05,\ 0.1, 0.2\} \times \{0.01,\ 0.05,\ 0.1,\ 0.2\}$ in terms of the performance of the algorithm.

We also used the DTLZ [65] and WFG [66] test suites to validate our approach. Both of these suites have become standards for judging the performance of multi- and many-objective optimization algorithms, with widespread use in recent research studies (e.g., [67–70]). Furthermore, a survey by Guo [71] stated that these two benchmarks were challenging enough to be used for the performance assessment of newly proposed MOEAs. In view of this, and as mentioned by many other authors, we consider both test suites pertinent.

The DTLZ and WFG problems are scalable in terms of the number of decision variables (n) and objective functions (m). These suites provide 16 unconstrained continuous problems (seven DTLZ problems plus nine WFG problems); jointly, they offer a wide range of properties in terms of their Pareto frontiers, being reasonably representative. Each problem was tested with five and 10 objective functions. In total, we tested 32 input instances, each of which was customized in regard to the parameters n (number of decision variables) and k (number of position-related variables) as follows:

- For DTLZ: $n = m + k + 1$, where $k = 5$ for DTLZ1, $k = 10$ for DTLZ2–6, and $k = 20$ for DTLZ7.
- For WFG: $k = 2(m - 1)$, and $n = 47$ if $m = 5$, or $n = 105$ if $m = 10$.

In addition, 20 preference systems were synthetically generated by choosing the priority objectives at random (with four priority objectives for $m = 10$, and two for $m = 5$). The parameters of the sigmoid functions were standardized as follows:

- Membership function for closeness, $\mu(c_i)$: We used a value of $\gamma = 0.1$, and α was isolated from Eq. (7) with $\mu(c_i) = 0.99$ at $\delta_i(x) = 0.01$. This setting models the situation where the DM hesitates (with a degree of truth of 0.5) over whether a distance of 10 % can be considered close to their aspiration point in the *i*th objective; in contrast, the DM is sure for a distance of 1 %.
- Membership function for farness, $\mu(e_i)$: We used a value of $\gamma' = 0.25$, and α' was isolated from Eq. (13), with $\mu(e_i) = 0.99$ at $\delta_i(x) = 0.5$. This setting models the case where the DM hesitates (with a degree of truth of 0.5) over whether a distance of 25 % can be considered far from their aspiration point in the *i*th objective; in contrast, the DM is sure with a distance of 50 %.

5.2. Illustrative example: analysis of a single run

We consider a hypothetical DM faced with DTLZ1, with 10 objective functions. This DM is interested in addressing the problem using our approach. Our algorithm performs 228 iterations using a colony with 220 ants (i.e., 50,160 objective function evaluations). The ACO algorithm will interact with the DM as follows:

- Phase 1: after iteration #45
- Phase 2: after iteration #91
- Phase 3: after iteration #136
- Phase 4: after iteration #182
- Presentation of the final prescription: after iteration #228

During the first 45 iterations, the optimization algorithm searches for a representative and well-distributed sample of the Pareto frontier (this part of our approach behaves as an *a posteriori* algorithm). An approximation of the reference points can then be presented to the DM (as the algorithm keeps the reference points updated after each iteration).

Since the DM wants to optimize all objective functions, the DM will pressure the algorithm to simultaneously find solutions as close as possible to the ideal point for all criteria. The DM has no reason to settle for less at this point. The DM then articulates the first fuzzy predicate as stated in Eq. (9) (Phase 1): "A solution is preferentially close if it is *very close* to the aspiration point across all criteria."

Fig. 4 tracks the objective function throughout the optimization algorithm, at each iteration. Here, $\mathscr{P}_1(\cdot)$ starts at 0.045994 (iteration #46) and finishes at 0.058684 (iteration #91). This suggests that an ideal solution cannot be reached. The DM should move forward to Phase 2 by relaxing the preference model, as stated in Eq. (11) (Phase 2): "A solution is preferentially close if it is *close* to the aspiration point across all criteria."

As shown in Fig. (4), $\mathscr{P}_2(\cdot)$ starts at 0.242248 (iteration #92) and finishes at 0.242258 (iteration #136). This highly stable behavior suggests that the algorithm cannot simultaneously find a solution closer to the reference point for all objectives. The DM should therefore contemplate the idea of prioritizing objectives during the third interaction. Here, the hypothetical DM has indicated objectives 1–4 as being the main priorities. This objective prioritization is taken to model the fuzzy predicate as stated in Eq. (14) (Phase 3): "A solution is preferentially close if it is *close* to the aspiration point for the priority objectives and is *not far* from the aspiration point for the non-priority ones."

In Fig. 4, $\mathscr{P}_3(\cdot)$ starts at 0.221728 (iteration #137) and finishes at 0.793497 (iteration #182). This fuzzy preference function leads the optimization algorithm to solutions that match the DM's preference more closely. During the fourth interaction, the DM may explore the possibility of obtaining a better compromise by updating the preference closeness model, and by indicating the most important objective function among the priority ones. Here, the hypothetical DM indicates objective 1 as being the most important one. This two-level objective prioritization allows us to apply Eq. (20) (Phase 4, Case 2), whose interpretation is: "A solution is preferentially close if it is *very close* to the aspiration point for the highest priority objective and is *close* for the rest of the priority objectives, and is *not too far* for the non-priority ones."

In Fig. 4, $\mathscr{P}_4(\cdot)$ starts at 0.913502 (iteration #183) and finishes at 0.928901 (iteration #228). It is plausible that the DM is satisfied, as the degree of truth is nearly 0.93. According to Picos [72], the average values of truth associated with the linguistic labels *false, almost false, more false than true, as false as true, more true than false, almost true*, and *true* are as follows:

- *false* = 0.033
- *almost false* = 0.172
- *more false than true* = 0.341
- *as false as true* = 0.493
- *more true than false* = 0.661
- *almost true* = 0.833
- *true* = 0.966

The example presented in this section represented only one of many runs of our approach. We have chosen these results because they are representative and exemplify a straightforward way in which the DM may discover their preference model by gaining an idea of the ranges of the objective functions for optimized solutions and prioritizing

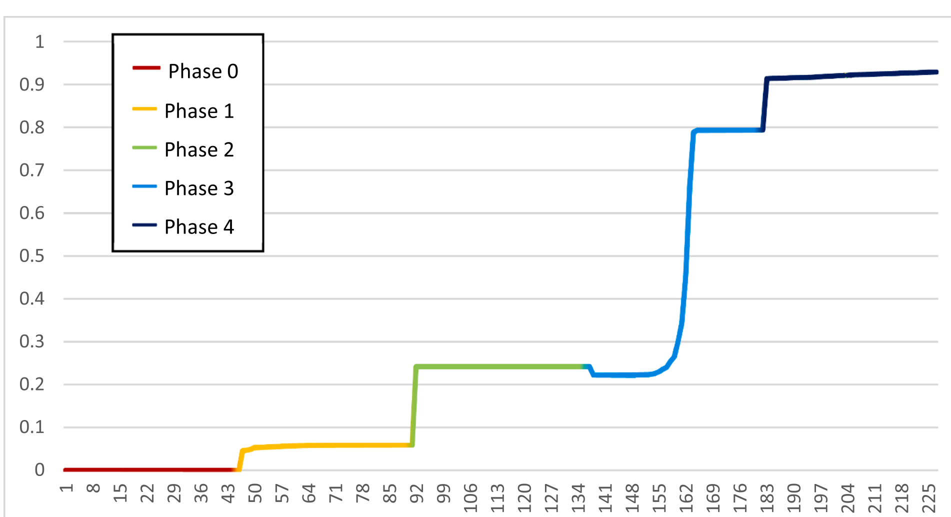


Fig. 4. Tracking of the fuzzy objective function.

objectives. The DM progressively learns about the problem and their own preference system.

5.3. Comparison with the state-of-the-art algorithms

RVEA-iGNG and TS-NSGA-II are two state-of-the-art algorithms for many-objective optimization. These two algorithms are competitive in terms of the most popular multi-objective indicators. Indeed, experimental results show that these have outperformed many evolutionary algorithms (cf. [63,64]). We therefore use RVEA-iGNG and TS-NSGA-II to validate whether our approach meets the standards for efficiency according to the state of the art.

The algorithms were configured to stop immediately after the iteration in which a total of 50,000 evaluations of the objective functions were reached. The parameter values of RVEA-iGNG and TS-NSGA-II were set as suggested by Liu et al. [63] and Ming et al. [64].

In addition, we calculated 1,000,000 Pareto-optimal points to measure convergence to the true Pareto frontier. The approximated RoI (A-RoI) is composed of the solutions with the best values in terms of the fuzzy predicate of Phase 4 (i.e., $\mathscr{P}_4^\star$ in Eq. (22)).

RVEA-iGNG and TS-NSGA-II were run 30 times per instance. For our ACO algorithm, we generated synthetic DMs by selecting different subsets of priority objectives (10 DMs for five-objective instances, and 20 DMs for 10-objective instances). Our algorithm was run 30 times per DM and per instance. We took the solution with the highest degree of truth from each single run as the best compromise offered by our proposal.

We assessed the quality of solutions based on the DM's satisfaction, as expressed by the fuzzy preference closeness. Classical multi-objective indicators measure the distribution, convergence, and extent of a sample of the true Pareto frontier. They are broadly accepted because *a posteriori* algorithms approximate the whole Pareto frontier. In contrast, *a priori* and interactive algorithms aim to approximate a subset of the Pareto frontier; in this case, the multi-objective indicators become misleading, as they do not consider the performance in terms of the preferred solutions (cf. [16]).

The metric used to assess the performance of a preference-based algorithm should therefore consider how closely the search algorithm follows the DM's preferences, i.e., whether the solutions actually reflect the DM's preferences [73]. Here, the best compromise solution is defined as the solution with the highest degree of truth for the fuzzy predicate that models the last expression of the DM's preferences.

Finally, we conducted Friedman non-parametric tests with Nemenyi post hoc analyses, using a confidence interval of 0.95, to assess the statistical significance of the results. Table 3 summarizes the results from the three algorithms for each instance. The average value of the fuzzy objective function is presented along with the gap regarding the A-RoI. All differences concerning ACO were significant.

The following observations can be made from Table 3:

- Our progressive approach using ACO consistently outperformed both state-of-the-art algorithms regarding the preference closeness-based model learned by the DM in the fourth phase.
- The gap between the results from the state-of-the-art algorithms and the A-RoI increased as the number of objectives rose.
- In contrast, the advantages of our approach became more pronounced as m increased, with a narrower gap for m=10 than for m=5.
- In the 10-objective problems, the average gap was 4.08 % for Case 1 and 5.16 % for Case 2. In the five-objective problems, the average gap was 6.02 % for Case 1 and 6.93 % for Case 2.
- The best performance (gap=0.16 %) was obtained for WFG8 with m=10. The Pareto frontier of WFG8 is non-separable, concave, markedly biased, and unimodal. According to Huband et al. [66], WFG8 is one of the most challenging problems in the WFG test suite.
- The worst performance (gap=10.87 %) was obtained for DTLZ1 with m=10. The Pareto frontier of DTLZ1 is separable, linear, non-biased, and multi-modal. It is interesting to note that the geometry of DTLZ1 shares few similarities with WFG8.

With the objective of analyzing the overall performance, we performed a Borda analysis of the three algorithms for each number of objectives. The algorithms were sorted by performance for each input instance, based on the Friedman test and the Nemenyi post-hoc analysis. The sum of their positions over the 16 problems describes the average performance of the metaheuristics.

Table 4 presents the Borda scores for each algorithm. Regardless of the preference closeness model used (Case 1 or Case 2) and the number of objectives, the ACO algorithm obtained a lower Borda score than the state-of-the-art algorithms, which indicates a better performance ranking. This difference becomes larger when $m = 10$; in this case, we suggest applying our approach, especially on many-objective optimization problems.

Lastly, we identified the number of DMs satisfied with the solution obtained by each algorithm. Again, we used the linguistic variables and their average degree of truth, as reported by Picos [72]. Fig. 5(*a*) shows the results for the problems with $m = 5$. We would like to emphasize the following points:

- There are 320 different DMs (two models, with 10 preference systems for each model, and 16 problems).
- The number of solutions close enough to the *true* value (0.966) in the A-RoI—the nearest approximation to the optimal solution we have—is 102 (see linguistic variable *true*). Our ACO algorithm identified 83, RVEA-iGNG identified 58, and TS-NSGA-II identified 61.
- The number of solutions that is close enough to *true* or *almost true* (0.833) in the A-RoI is 233. Our algorithm gave 225, RVEA-iGNG gave 218, and TS-NSGA-II gave 216.
- There are several cases in which the DM could always be dissatisfied, even with solutions in the A-RoI. These are the 26 cases where the best value reached was *as false as true.* The DM should settle for the 'least bad' solution in such a case.
- Lastly, our approach gave the lowest number of dissatisfied DMs (see the linguistic variable *more false than true*).

Similarly, Fig. 5(*b*) presents the results for the problems with $m = 10$. The following points summarize them:

- There are 640 different DMs (two models, with 20 preference systems for each model, and 16 problems).
- The number of solutions that were close enough to *true* in the A-RoI was 192. Our ACO algorithm identified 160, RVEA-iGNG identified 122, and TS-NSGA-II identified 109.
- The number of solutions that were close enough to *true* or *almost true* in the A-RoI was 483. Our algorithm gave 467, RVEA-iGNG gave 443, and TS-NSGA-II gave 454.
- There are several instances in which the DM could always be dissatisfied, even with the solutions in the A-RoI. These are the 39 cases where the best value reached was *as false as true* or *more false than true.* Even here, our algorithm presented the lowest number of dissatisfied DMs.

It is interesting that some preference systems never reached satisfactory solutions, regardless of the test suite and the number of objectives used. We hypothesize that this behavior occurs because of the geometry of the Pareto frontiers, which have challenging properties in these synthetic problems. In such cases (26 for $m = 5$ and 39 for $m = 10$) our approach failed to find a satisfactory solution because when some objectives were too close to their aspiration values, others were markedly degraded. In these cases, sufficiently high values of the predicate

Table 3
Results on the DTLZ and WFG test suites.

Problem	m	Algorithm	Phase 1	Phase 2	Phase 3	Phase 4		Difference from the A-RoI	
						Case 1	Case 2	Case 1	Case 2
DTLZ1	5	ACO	0.110102	0.103464	0.785457	0.901270	0.917712	6.74 %	4.72 %
		TS-NSGA-II	0.105662	0.176790	0.727323	0.893607	0.896709	7.54 %	6.90 %
		RVEA-iGNG	0.098372	0.234386	0.740302	0.884096	0.889010	8.52 %	7.70 %
	10	ACO	0.099305	0.208883	0.725264	0.903861	0.848183	4.28 %	10.87 %
		TS-NSGA-II	0.095156	0.201286	0.447415	0.838135	0.793396	11.24 %	16.63 %
		RVEA-iGNG	0.092033	0.178582	0.642187	0.862166	0.788085	8.69 %	17.19 %
DTLZ2	5	ACO	0.181864	0.199431	0.747739	0.924796	0.901915	7.27 %	9.66 %
		TS-NSGA-II	0.181249	0.170385	0.679505	0.896790	0.861169	10.08 %	13.74 %
		RVEA-iGNG	0.174383	0.146559	0.700901	0.903779	0.875128	9.38 %	12.34 %
	10	ACO	0.155352	0.308631	0.742551	0.903411	0.930454	4.59 %	4.93 %
		TS-NSGA-II	0.152301	0.177538	0.367296	0.846377	0.837768	10.61 %	14.40 %
		RVEA-iGNG	0.140565	0.230764	0.616245	0.836054	0.856920	11.70 %	12.45 %
DTLZ3	5	ACO	0.138235	0.139541	0.763064	0.917778	0.904543	4.32 %	6.55 %
		TS-NSGA-II	0.136294	0.179473	0.695183	0.909073	0.881909	5.23 %	8.89 %
		RVEA-iGNG	0.125454	0.096520	0.735037	0.896471	0.871371	6.54 %	9.98 %
	10	ACO	0.073380	0.087701	0.745427	0.909423	0.949628	5.55 %	0.62 %
		TS-NSGA-II	0.071369	0.193558	0.613508	0.851310	0.861698	11.59 %	9.82 %
		RVEA-iGNG	0.068215	0.086884	0.431469	0.842870	0.882599	12.47 %	7.64 %
DTLZ4	5	ACO	0.142543	0.143110	0.794635	0.918313	0.918138	6.95 %	6.61 %
		TS-NSGA-II	0.138321	0.286597	0.753605	0.888618	0.914050	9.96 %	7.03 %
		RVEA-iGNG	0.128862	0.154125	0.769606	0.911186	0.898028	7.67 %	8.66 %
	10	ACO	0.070818	0.131675	0.756206	0.920976	0.813677	6.75 %	7.15 %
		TS-NSGA-II	0.068464	0.075239	0.680873	0.869443	0.765887	11.97 %	12.60 %
		RVEA-iGNG	0.063995	0.113688	0.612168	0.882636	0.770321	10.63 %	12.10 %
DTLZ5	5	ACO	0.250419	0.330873	0.776172	0.918339	0.904918	4.60 %	6.05 %
		TS-NSGA-II	0.244907	0.358680	0.747716	0.875526	0.866263	9.04 %	10.06 %
		RVEA-iGNG	0.225643	0.260566	0.755044	0.902942	0.891952	6.20 %	7.40 %
	10	ACO	0.051370	0.114657	0.773169	0.924775	0.779468	0.39 %	6.43 %
		TS-NSGA-II	0.047716	0.119591	0.545378	0.852345	0.710684	8.19 %	14.68 %
		RVEA-iGNG	0.046072	0.149417	0.529217	0.886794	0.721904	4.48 %	13.34 %
DTLZ6	5	ACO	0.236507	0.257975	0.745913	0.900374	0.909695	5.47 %	6.88 %
		TS-NSGA-II	0.231107	0.282180	0.715502	0.888840	0.867589	6.68 %	11.19 %
		RVEA-iGNG	0.220663	0.175452	0.693381	0.868028	0.880276	8.86 %	9.90 %
	10	ACO	0.038904	0.149552	0.820332	0.922999	0.942863	3.36 %	4.25 %
		TS-NSGA-II	0.038358	0.130817	0.403251	0.857518	0.872323	10.22 %	11.41 %
		RVEA-iGNG	0.036856	0.166407	0.668813	0.880616	0.903945	7.80 %	8.20 %
DTLZ7	5	ACO	0.100752	0.231556	0.798398	0.923033	0.909126	6.79 %	7.27 %
		TS-NSGA-II	0.093553	0.090382	0.720742	0.886701	0.874132	10.46 %	10.84 %
		RVEA-iGNG	0.089104	0.186374	0.770985	0.892096	0.873846	9.91 %	10.87 %
	10	ACO	0.108504	0.254808	0.809284	0.909228	0.851647	8.96 %	9.21 %
		TS-NSGA-II	0.105471	0.107499	0.517919	0.844483	0.787622	15.45 %	16.04 %
		RVEA-iGNG	0.100294	0.180270	0.446415	0.860882	0.795272	13.81 %	15.22 %
WFG1	5	ACO	0.201719	0.256567	0.750660	0.915675	0.908756	6.15 %	5.21 %
		TS-NSGA-II	0.184690	0.219607	0.697421	0.907087	0.900937	7.03 %	6.03 %
		RVEA-iGNG	0.175584	0.214598	0.698185	0.898134	0.900917	7.94 %	6.03 %
	10	ACO	0.126461	0.211747	0.821162	0.916788	0.887858	3.99 %	3.30 %
		TS-NSGA-II	0.124402	0.261306	0.633904	0.844492	0.807712	11.56 %	12.03 %
		RVEA-iGNG	0.118583	0.136465	0.470676	0.872762	0.824098	8.60 %	10.25 %
WFG2	5	ACO	0.167920	0.280483	0.789036	0.912564	0.901191	5.37 %	8.71 %
		TS-NSGA-II	0.154588	0.193252	0.732572	0.900666	0.860669	6.61 %	12.82 %
		RVEA-iGNG	0.147804	0.295434	0.759042	0.881333	0.896486	8.61 %	9.19 %
	10	ACO	0.120217	0.130268	0.785727	0.903134	0.888626	3.49 %	6.08 %
		TS-NSGA-II	0.115238	0.159556	0.723467	0.819892	0.821597	12.38 %	13.17 %
		RVEA-iGNG	0.108874	0.116427	0.645810	0.848627	0.823045	9.31 %	13.01 %
WFG3	5	ACO	0.115687	0.150643	0.741815	0.905656	0.913974	5.10 %	8.28 %
		TS-NSGA-II	0.106514	0.259662	0.721609	0.884389	0.897338	7.33 %	9.95 %
		RVEA-iGNG	0.100563	0.151243	0.690023	0.870800	0.888957	8.75 %	10.79 %
	10	ACO	0.029195	0.139902	0.815062	0.907819	0.901954	6.46 %	6.29 %
		TS-NSGA-II	0.028149	0.095181	0.771960	0.837362	0.827121	13.72 %	14.07 %
		RVEA-iGNG	0.026384	0.172169	0.641043	0.839632	0.861838	13.48 %	10.46 %
WFG4	5	ACO	0.113149	0.165574	0.784127	0.906465	0.901114	7.31 %	7.44 %
		TS-NSGA-II	0.111610	0.132423	0.765237	0.889373	0.894862	9.05 %	8.09 %
		RVEA-iGNG	0.106728	0.119569	0.731221	0.895481	0.881569	8.43 %	9.45 %
	10	ACO	0.145812	0.304179	0.819068	0.916575	0.836359	3.22 %	6.66 %
		TS-NSGA-II	0.132492	0.143312	0.786423	0.839503	0.789055	11.35 %	11.94 %
		RVEA-iGNG	0.127950	0.187754	0.495228	0.876051	0.784886	7.49 %	12.40 %
WFG5	5	ACO	0.159546	0.130361	0.750703	0.916806	0.914033	4.95 %	5.18 %
		TS-NSGA-II	0.147411	0.170787	0.713203	0.882329	0.888987	8.52 %	7.78 %
		RVEA-iGNG	0.142792	0.157651	0.698591	0.899510	0.891870	6.74 %	7.48 %
	10	ACO	0.064880	0.200308	0.728459	0.906368	0.940817	0.64 %	4.26 %
		TS-NSGA-II	0.064495	0.069087	0.429588	0.857782	0.867184	5.97 %	11.76 %
		RVEA-iGNG	0.062115	0.179906	0.569676	0.870044	0.878768	4.63 %	10.58 %
WFG6	5	ACO	0.105206	0.255472	0.753491	0.919482	0.900959	6.40 %	9.65 %

(continued on next page)

Table 3 (*continued*)

Problem	*m*	Algorithm	Phase 1	Phase 2	Phase 3	Phase 4		Difference from the A-RoI	
						Case 1	Case 2	Case 1	Case 2
		TS-NSGA-II	0.101402	0.241643	0.727706	0.903930	0.868435	7.98 %	12.91 %
		RVEA-iGNG	0.097014	0.079093	0.699615	0.907734	0.867651	7.59 %	12.99 %
	10	ACO	0.064435	0.204131	0.771005	0.912607	0.794474	1.16 %	3.90 %
		TS-NSGA-II	0.062439	0.219754	0.632818	0.858602	0.744035	7.01 %	10.00 %
		RVEA-iGNG	0.059125	0.111928	0.583781	0.840155	0.736295	9.01 %	10.93 %
WFG7	5	ACO	0.160423	0.262063	0.735787	0.905947	0.914411	7.21 %	5.10 %
		TS-NSGA-II	0.151056	0.277552	0.688242	0.867766	0.895233	11.12 %	7.09 %
		RVEA-iGNG	0.140182	0.233682	0.712984	0.881815	0.909802	9.68 %	5.58 %
	10	ACO	0.075378	0.197683	0.792946	0.907370	0.916117	7.81 %	4.51 %
		TS-NSGA-II	0.070024	0.171670	0.517355	0.829746	0.866257	15.69 %	9.70 %
		RVEA-iGNG	0.066550	0.175411	0.539439	0.851034	0.862572	13.53 %	10.09 %
WFG8	5	ACO	0.170916	0.293011	0.828364	0.923281	0.923013	5.94 %	6.88 %
		TS-NSGA-II	0.155915	0.241798	0.762306	0.911826	0.878042	7.10 %	11.41 %
		RVEA-iGNG	0.150382	0.291036	0.795228	0.886555	0.905146	9.68 %	8.68 %
	10	ACO	0.111354	0.118827	0.748182	0.916491	0.942364	0.16 %	3.84 %
		TS-NSGA-II	0.110706	0.183468	0.551997	0.836830	0.861967	8.84 %	12.04 %
		RVEA-iGNG	0.107372	0.144548	0.497092	0.866319	0.870809	5.63 %	11.14 %
WFG9	5	ACO	0.106349	0.097838	0.799720	0.918498	0.912167	5.71 %	6.71 %
		TS-NSGA-II	0.096188	0.122616	0.732340	0.896644	0.881168	7.95 %	9.88 %
		RVEA-iGNG	0.089828	0.128899	0.750328	0.906352	0.906981	6.96 %	7.24 %
	10	ACO	0.061005	0.131634	0.824535	0.917706	0.924207	4.41 %	0.33 %
		TS-NSGA-II	0.055167	0.125995	0.796374	0.852041	0.872956	11.25 %	5.85 %
		RVEA-iGNG	0.050850	0.173288	0.518876	0.852925	0.883989	11.15 %	4.66 %

Table 4
Borda scores for the three algorithms.

Model	*m*	Algorithm	Borda score	Model	*m*	Algorithm	Borda score
Case 1	5	ACO	16	Case 2	5	ACO	16
		RVEA-iGNG	38			RVEA-iGNG	39
		TS-NSGA-II	42			TS-NSGA-II	41
	10	ACO	16		10	ACO	16
		RVEA-iGNG	40			RVEA-iGNG	40
		TS-NSGA-II	40			TS-NSGA-II	40

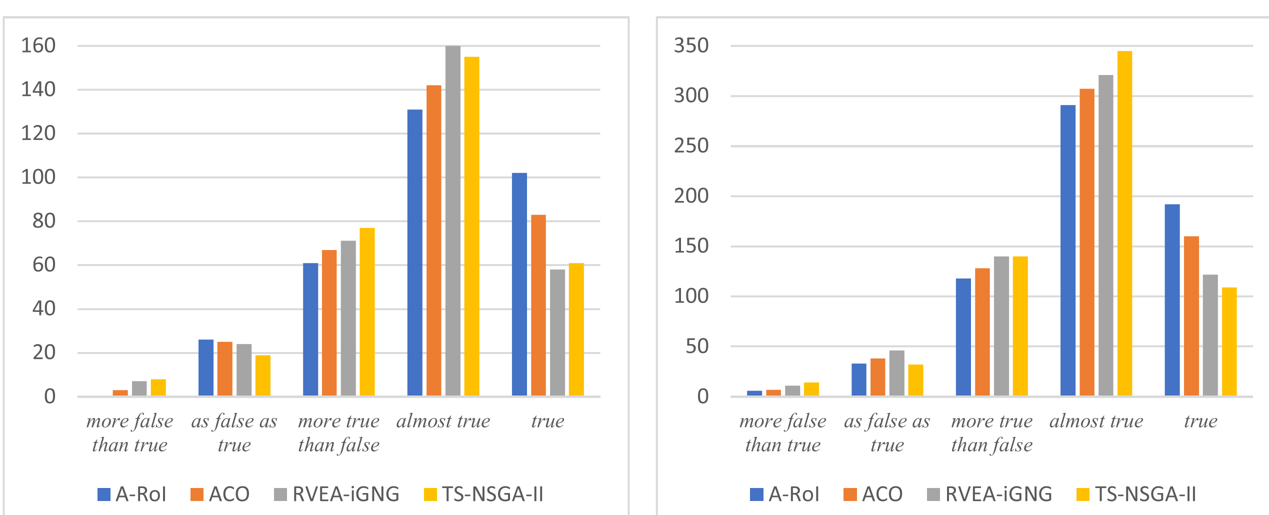
about preference closeness were unfeasible, so the preference model could not find a satisfactory trade-off. In such cases, the DM should relax their notions of closeness (for the priority objectives) and farness (for the non-priority objectives). This task could be performed by updating the parameters of the sigmoid functions $\mu(a_i)$ and $\mu(c_i)$ (Eqs. (6) and (13)).

As a last remark on Fig. 5, we would like to emphasize that our algorithm's output was distributed more similarly to the A-RoI than the reference algorithms for both $m = 5$ and $m = 10$, indicating that it meets the desired quality standard according to the scientific literature.

6. Conclusions and directions for future research

This paper has presented a preference model based on compensatory fuzzy logic, which can be embedded in search algorithms to solve MaOPs. The model makes several innovative contributions:

- The best compromise solution is described as the preferentially closest to an aspiration point. As a side benefit, this definition is reasonably interpretable, as it is expressed in almost natural language.



(a) Five-objective problems ***(b)*** 10-objective problems

Fig. 5. Number of DMs that would fall into each category.

- To the best of our knowledge, this model is the first to define the preference closeness through fuzzy logic.
- Our scheme allows DMs to discover their system of preferences in a progressive fashion. During each interaction, the DM understands the preference closeness better by perusing the reference vectors and the optimized solutions. From the perspective of hybrid-augmented intelligence, this feature makes our approach one of the most integral ways of addressing MaOPs.
- The model is highly customizable to the DM's preferences, and supports any fuzzy predicate expressed in terms of closeness to the reference points and priorities of the objectives.
- The model demands minimum cognitive effort; the DM does not need to make any comparisons or to rank solutions, which would be particularly challenging in the presence of many criteria. The DM is only required to express their current notion of preference closeness in terms of nearness to their aspiration point and the priority attached to each objective.
- Vagueness, hesitation, and imprecision can be handled via compensatory fuzzy logic, which adds value to the proposed model.

In this paper, the preference model was instantiated in two cases. In the first case, we considered two priority levels attached to the objectives, and in the second, we considered three. Both model instances were embedded in an ACO algorithm and validated on 16 unconstrained optimization problems. These problems formed part of the WFG and DTLZ test suites, which are challenging and representative sets in the field of many-objective optimization. Compared to state-of-the-art algorithms (specifically, RVEA-iGNG and TS-NSGA-II), our approach showed better convergence to the best compromise solution. The results were statistically tested, and were particularly encouraging for 10-objective problems.

In future research, we will conduct broader experimentation to assess the synergy of this approach with more metaheuristic algorithms on different benchmark problems. Lastly, it would also be interesting to elicit the parameters of the sigmoid functions during the interactions with the DM, especially in those preference systems where a satisfactory solution was not found (26 for the five-objective problems and 39 for the 10-objective problems). The DM could learn not only a definition of the preference closeness but also the definitions of the linguistic variables "close to" and "far from" for each objective regarding the reference points.

Appendix: List of acronyms

ACO	Ant Colony Optimization
$ACO_{\mathbb{R}}$	Ant Colony Optimization for Continuous Domains
AI	Artificial Intelligence
DM	Decision Maker
DTLZ	Deb, Thiele, Laumanns, and Zitzler
GMCFL	Geometric Mean-Based Compensatory Fuzzy Logic
HAI	Hybrid-Augmented Intelligence
MaOP	Many-Objective Optimization Problem
MOEA	Multi-objective Evolutionary Algorithm
MOP	Multi-objective Optimization Problem
RoI	Region of Interest
RVEA-iGNG	Reference-Vector-Guided Evolutionary Algorithm with Improved Growing Neural Gas
TS-NSGA-II	Two-Stage Non-dominated Sorting Genetic Algorithm II
WFG	Walking Fish Group

CRediT authorship contribution statement

Eduardo Fernandez: Supervision, Methodology, Conceptualization. **Gilberto Rivera:** Writing – original draft, Validation, Software. **Laura Cruz-Reyes:** Project administration, Methodology, Formal analysis. **Rafael A. Espin-Andrade:** Investigation, Formal analysis, Conceptualization. **Claudia G. Gomez-Santillan:** Writing – original draft, Software, Investigation. **Nelson Rangel-Valdez:** Validation, Resources, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Further reading

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