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On solutions of PDEs by using algebras

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The components of complex analytic functions define solutions for the Laplace's equation, and in a simply connected domain, each solution of this equation is the first component of a complex analytic function. In this paper, we generalize this result; for each PDE of the form $Au_{xx}+Bu_{xy}+Cu_{yy}=0$, and for each affine planar vector field φ, we give an algebra $\mathbb{A}$ with unit $e=e_1$, with respect to which the components of all functions of the form $\mathcal{L}\circ\varphi$ are all the solutions for this PDE, where $\mathcal{L}$ is differentiable in the sense of Lorch with respect to $\mathbb{A}$. Solutions are also constructed for the following equations: $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$, 3rd-order PDEs, and 4th-order PDEs; among these are the bi-harmonic, the bi-wave, and the bi-telegraph equations.

KEYWORDS

differentiation theory, heat equation, Laplace's equation, matrix exponential function, partial differential equations (PDEs), wave equation

MSC CLASSIFICATION

15A16; 30G35; 35A25; 58C20

1 | INTRODUCTION

An algebraic-analytical approach applied to families of PDEs is presented in this paper. The corresponding ideas were given by Ketchum[1] where differentiability in the sense of Lorch with respect to associative commutative algebras with unit e is used; see other studies.[2–4] Instead, we propose the use of pre-twisted differentiability with respect to a family of algebras with unit $e=e_1$; see preprint arXiv 1805.10524. We obtain a complete solution of each PDE of the family

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0, \tag{1}$$

with constant coefficients, and we construct solutions for the class of PDEs of the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0, \tag{2}$$

with constant coefficients, which includes the one-dimensional heat equation

$$\alpha u_{xx}-u_t=0, \tag{3}$$

where the variable y is changed to t. The subclass of PDEs having the form (1) includes Laplace's and wave equations, which are given by

$$u_{xx}+u_{yy}=0, \qquad c^2u_{xx}-u_{tt}=0, \tag{4}$$

respectively, where in the last equation, the variable y is changed to t.

When proposing a solution of the form $w = e^{ax+by}$ for (2), it is concluded that (a, b) must be a solution of the algebraic quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The set of solutions of the this equation defines a conic, unless it is degenerate. Given a PDE like (2) and a vector field

$$\varphi(x, y) = (ax + by + k, cx + dy + l), \tag{5}$$

with $Ac^2 + Bcd + Cd^2 \neq 0$, in this paper, we found an algebra $\mathbb{A}$ with respect to which the components of the exponential function

$$\mathcal{E}(x, y) = e^{\varphi(x,y)}, \tag{6}$$

define solutions of (2). If $D = E = 0$ in (2), similar results are obtained by using sine, cosine, hyperbolic sine, and hyperbolic cosine functions instead of the exponential function. For the first-order case, that is, when A, B, and C are zero, we found conditions such that the components of the exponential function (6) define solutions for all the two-dimensional algebras.

The components of complex analytic functions are harmonic functions, and in a simply connected domain, each harmonic function is the first component of a complex analytic function. This result has been generalized in Theorems 4.1 and 5.1; for each PDE (1) and for each affine planar vector field φ with $Ac^2 + Bcd + Cd^2 \neq 0$, we found an associative and commutative two-dimensional algebra $\mathbb{A}$ with unit $e = e_1$ (see Section 2.1), with respect to which the components of all $\varphi\mathbb{A}$-differentiable functions (see Section 2.2) are solutions for this PDE, and on simply connected regions, we show that the solutions of (1) are components of these $\varphi\mathbb{A}$-differentiable functions.

All the theorems, propositions, corollaries, and examples included in this paper are presented for the first time.

In Section 2, we introduce the algebras $\mathbb{A}$ that we will use, the $\varphi\mathbb{A}$-differentiability, and its generalized Cauchy–Riemann equations. In Section 3, given a 2nd-order PDE, and an affine planar vector field, we give an algebra with respect to which the components of the exponential function e^{φ} define solutions of the PDE, and we give these solutions explicitly. Also, we use the sine, cosine, hyperbolic sine, and hyperbolic cosine functions instead the exponential function, for constructing solutions of 2nd-order PDEs. Moreover, for each PDE of the form (1), in Section 4, we show families of pre-twisted differentiable functions whose components are solutions, and in Section 5, we show that each solution of these PDEs is a component of a $\varphi\mathbb{A}$-differentiable function. In Section 6, 3rd-order PDEs are considered, and in Section 7, three 4th PDEs are considered; the bi-harmonic, the bi-wave, and the bi-telegraph equations. Section 8 contains the case of first-order PDEs. The conclusions are contained in Section 9. The next two sections are about interest statement and acknowledgment.

2 | PRE-TWISTED DIFFERENTIABILITY

2.1 | Algebras $\mathbb{A}_1(p_1, p_2)$

We call to a $\mathbb{R}$-linear space $\mathbb{L}$ *an algebra* if it is endowed with a bilinear product $\mathbb{L} \times \mathbb{L} \to \mathbb{L}$ denoted by $(u, v) \mapsto uv$, which is associative and commutative $u(vw) = (uv)w$ and $uv = vu$ for all $u, v, w \in \mathbb{L}$; furthermore, there exists a unit $e \in \mathbb{L}$, which satisfies $eu = u$ for all $u \in \mathbb{L}$. An element $u \in \mathbb{L}$ is called *regular* if there exists $u^{-1} \in \mathbb{L}$ called *the inverse* of u such that $u^{-1}u = e$. We also use the notation e/u for u^{-1}. If $u \in \mathbb{L}$ is not regular, then u is called *singular*. $\mathbb{L}^*$ denotes the set of all the regular elements of $\mathbb{L}$. If $u, v \in \mathbb{L}$ and v is regular, the quotient u/v means uv^{-1}.

An *algebra* $\mathbb{A}$ will be an algebra if $\mathbb{L} = \mathbb{R}^2$, and an *algebra* $\mathbb{M}$ will be an algebra if $\mathbb{L}$ is a two dimensional matrix algebra in the space of matrices $M(2, \mathbb{R})$, where the algebra product corresponds to the matrix product. We say that two matrix algebras $\mathbb{M}_1$, and $\mathbb{M}_2$ are *conjugated* if there exists an invertible matrix T such that $\mathbb{M}_1 = T\mathbb{M}_2T^{-1}$.

The $\mathbb{A}$-product between the elements of the canonical basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ is given by $e_ie_j = \sum_{k=1}^{2} c_{ijk}e_k$, where $c_{ijk} \in \mathbb{R}$ for $i, j, k \in \{1, 2\}$ are called *structure constants* of $\mathbb{A}$. The *first fundamental representation* of $\mathbb{A}$ is the injective linear homomorphism $R : \mathbb{A} \to M(2, \mathbb{R})$ defined by $R : e_i \mapsto R_i$, where R_i is the matrix with $[R_i]_{jk} = c_{ikj}$, for $i = 1, 2$.

The linear space $\mathbb{R}^2$ endowed with the product

$$\begin{array}{c|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & p_1e_1 + p_2e_2 \end{array} \tag{7}$$

is an algebra $\mathbb{A}$ which we denote by $\mathbb{A}_1(p_1, p_2)$; see Frías-Armenta and López-González.[5] These algebras are associative, commutative, and have unit $e = e_1$; see also Pierce.[6] The *first fundamental representation* of $\mathbb{A}_1(p_1, p_2)$ is defined by

$$R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix}. \tag{8}$$

The first fundamental representation R of $\mathbb{A}$ allows us to perform arithmetic operations, and powers among others, in $\mathbb{A}$, by performing them in $\mathbb{M} = R(\mathbb{A})$.

2.2 | $\varphi\mathbb{A}$-differentiability

The pre-twisted differentiability is defined in preprint arXiv 1805.10524; this definition is closely related with the differentiability in the sense of Lorch; see Lorch.[2] Let $\mathbb{A}$ be an algebra and φ a differentiable planar vector field in the usual sense. We say that a planar vector field $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable (pre-twisted differentiable) if $\mathcal{F}$ is differentiable in the usual sense, and if there exists a planar vector field $\mathcal{F}'_\varphi$, which we call $\mathbb{A}$-*derivative* of $\mathcal{F}$, such that

$$d\mathcal{F}_p = \mathcal{F}'_\varphi(p)d\varphi_p, \qquad p = (x, y), \tag{9}$$

where $\mathcal{F}'_\varphi(p)d\varphi_p(v)$ denotes the $\mathbb{A}$-product of $\mathcal{F}'_\varphi(p)$ and $\varphi_p(v)$ for every vector v in $\mathbb{R}^2$. In the same way, we say that $\mathcal{F}$ has a *second-order* $\varphi\mathbb{A}$*-derivative* $\mathcal{F}''_\varphi$ if $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable, $\mathcal{F}'_\varphi$ is differentiable in the usual sense, and $\mathcal{F}''_\varphi$ is a planar vector field, such that

$$d(\mathcal{F}'_\varphi)_p = \mathcal{F}''_\varphi(p)d\varphi_p, \qquad p = (x, y). \tag{10}$$

In this way, we define the *n-order* $\varphi\mathbb{A}$*-derivative* $\mathcal{F}^{(n)}_\varphi$ for $n \in \mathbb{N}$.

A $\varphi\mathbb{A}$*-polynomial function* $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ is defined by

$$\mathcal{P}(\xi) = c_0 + c_1\varphi(\xi) + c_2(\varphi(\xi))^2 + \dots + c_m(\varphi(\xi))^m, \tag{11}$$

where $c_0, c_1, \dots, c_m \in \mathbb{A}$ are constants, $\xi = (x, y)$ is $\mathbb{A}$-variable, and the products involved in $c_k(\varphi(\xi))^k$ for $k \in \{1, 2, \dots, m\}$ are defined with respect to $\mathbb{A}$. In the same way, *exponential*, *trigonometric*, and *hyperbolic* $\varphi\mathbb{A}$*-functions* are defined. If $\mathcal{P}$ and $\mathcal{Q}$ are $\varphi\mathbb{A}$-polynomial functions, the $\varphi\mathbb{A}$*-rational function* $\mathcal{P}/\mathcal{Q}$ is defined on the set $\mathcal{Q}^{-1}(\mathbb{A}^*)$. All these functions have n-order $\varphi\mathbb{A}$-derivatives for $n \in \mathbb{N}$, and the usual rules of differentiation are satisfied for this differentiability.

When we are working with n-order PDEs, we will assume that all $\varphi\mathbb{A}$-differentiable functions have n-order $\varphi\mathbb{A}$-derivatives. That is, we assume that the $\varphi\mathbb{A}$-differentiable functions have derivatives of the necessary orders.

2.3 | Partial derivatives of $\mathcal{F}$

Consider a $\varphi\mathbb{A}$-differentiable function $\mathcal{F}$. Thus, the first partial derivatives $\mathcal{F}_x$ and $\mathcal{F}_y$ are expressed by

$$\mathcal{F}_x = \mathcal{F}'_\varphi\varphi_x, \qquad \mathcal{F}_y = \mathcal{F}'_\varphi\varphi_y, \tag{12}$$

the second ones $\mathcal{F}_{xx}$, $\mathcal{F}_{xy}$, and $\mathcal{F}_{yy}$ by

$$\mathcal{F}_{xx} = \mathcal{F}''_\varphi\varphi_x^2 + \mathcal{F}'_\varphi\varphi_{xx}, \qquad \mathcal{F}_{xy} = \mathcal{F}''_\varphi\varphi_x\varphi_y + \mathcal{F}'_\varphi\varphi_{xy}, \qquad \mathcal{F}_{yy} = \mathcal{F}''_\varphi\varphi_y^2 + \mathcal{F}'_\varphi\varphi_{yy}. \tag{13}$$

Since φ is an affine vector field, its second partial derivatives are zero, then the second partial derivatives given in (13) become

$$\mathcal{F}_{xx} = \mathcal{F}''_\varphi\varphi_x^2, \qquad \mathcal{F}_{xy} = \mathcal{F}''_\varphi\varphi_x\varphi_y, \qquad \mathcal{F}_{yy} = \mathcal{F}''_\varphi\varphi_y^2. \tag{14}$$

From the product of $\mathbb{A} = \mathbb{A}^1(p_1, p_2)$, and the form of φ in (5), we have

$$\begin{aligned} \varphi_x^2 &= (a^2 + p_1c^2, 2ac + p_2c^2), \\ \varphi_x\varphi_y &= (ab + p_1cd, ad + bc + p_2cd), \\ \varphi_y^2 &= (b^2 + p_1d^2, 2bd + p_2d^2), \end{aligned} \tag{15}$$

which can be calculated by using the first fundamental representation of $\mathbb{A}$.

2.4 | $\varphi\mathbb{A}$ Cauchy–Riemann equations

By using $\mathcal{F}_x$ and $\mathcal{F}_y$ given in (12), we obtain

$$\varphi_y\mathcal{F}_x = \varphi_x\mathcal{F}_y.$$

For $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, and φ given by (5), this equation gives the $\varphi\mathbb{A}$ *Cauchy-Riemann equations*

$$\begin{aligned} bf_x + p_1dg_x &= af_y + p_1cg_y, \\ df_x + (b + p_2d)g_x &= cf_y + (a + p_2c)g_y. \end{aligned} \tag{16}$$

So, the components of $\varphi\mathbb{A}$-differentiable functions $\mathcal{F} = (f, g)$ satisfy (16). Conversely, if f, g satisfy (16), and φ is an isomorphism, then $\mathcal{F} = (f, g)$ is $\varphi\mathbb{A}$-differentiable; see preprint arXiv 1805.10524.

3 | SOLUTIONS DEFINED BY $\mathcal{E} = e^{\varphi}$

3.1 | Looking for the algebra

In the following theorem, we found an algebra $\mathbb{A}$ with respect to which the components of the exponential function $\mathcal{E} = e^{\varphi}$ define solutions of the PDE (2).

Theorem 3.1. *Consider the PDE (2) and the vector field φ given in (5). Suppose that $Ac^2 + Bcd + Cd^2 \neq 0$. Thus, for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ with parameters p_1 and p_2 given by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 + Da + Eb + F}{Ac^2 + Bcd + Cd^2}, \tag{17}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd + Dc + Ed}{Ac^2 + Bcd + Cd^2}, \tag{18}$$

the components f and g of the exponential function (6) defined with respect to $\mathbb{A}$, are solutions of the PDE (2).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$. The equalities (17) and (18) are equivalent to

$$\begin{aligned} A(a^2 + p_1c^2) + B(ab + p_1cd) + C(b^2 + p_1d^2) + Da + Eb + F = 0, \\ A(2ac + p_2c^2) + B(ad + bc + p_2cd) + C(2bd + p_2d^2) + Dc + Ed = 0, \end{aligned} \tag{19}$$

respectively. From (15) and (19), it can be obtained

$$A\varphi_x^2 + B\varphi_x\varphi_y + C\varphi_y^2 + D\varphi_x + E\varphi_y + F(1, 0) = 0. \tag{20}$$

Since $\mathcal{E} = e^{\varphi}$ with respect to the product of $\mathbb{A}$, we have $\mathcal{E} = \mathcal{E}'_{\varphi} = \mathcal{E}''_{\varphi}$. From this and the equalities (12) and (14), by multiplying $\mathcal{E}$ with respect to $\mathbb{A}$, we get

$$A\mathcal{E}_{xx} + B\mathcal{E}_{xy} + C\mathcal{E}_{yy} + D\mathcal{E}_x + E\mathcal{E}_y + F\mathcal{E} = 0. \tag{21}$$

Since $\mathcal{E} = (f, g)$, we have that f and g are solutions for (2). □

By using the first fundamental representation R defined in Section 2.1, expressions for f and g can be obtained, as we see in the following example.

Example 3.1 Consider the PDE (2) with $A = 1, B = 2, C = 3, D = 4, E = 5$, and $F = 6$. If $\varphi(x, y) = (0, x + y)$, then $p_1 = -1, p_2 = -3/2$. Using the first fundamental representation R, we can find the components f and g of e^{φ}, so

$$f(x,y) = \frac{7\cos\left(\frac{\sqrt{7}(x+y)}{4}\right) + 3\sqrt{7}\sin\left(\frac{\sqrt{7}(x+y)}{4}\right)}{7e^{\frac{3(x+y)}{4}}}, \tag{22}$$

and

$$g(x,y) = \frac{4\sqrt{7}\sin\left(\frac{\sqrt{7}(x+y)}{4}\right)}{7e^{\frac{3(x+y)}{4}}}. \tag{23}$$

By Theorem 3.1, f and g are solutions for (2). If $\varphi(x, y) = (0, x)$, then $p_1 = -6, p_2 = -4$,

$$f(x,y) = e^{-2x}\left(\cos\left(\sqrt{2}x\right) + \sqrt{2}\sin\left(\sqrt{2}x\right)\right), \tag{24}$$

and

$$g(x,y) = \frac{\sqrt{2}}{2}e^{-2x}\sin\left(\sqrt{2}x\right). \tag{25}$$

By Theorem 3.1, f and g are solutions for (2).

3.2 | Expression for $\mathcal{E} = e^{\varphi}$

In this section, we will use the Theorem 3.1 to give explicit solutions for PDEs of type (2). By means of the first fundamental representation R defined in the Section 2.1, the operations of $\mathbb{A}$ correspond to the matrix operations of $R(\mathbb{A})$. Although the computation of the of matrix exponentials is well-known and commonly used in differential equations, this section makes the paper more complete since solutions for (2) can be directly obtained from the expressions for f and g given here.

To each matrix in its normal form, a matrix algebra can be associated; see Alvarez-Parrilla et al.[7]; in the case of 2-by-2 real matrices, we have three types of algebras that correspond to the three types of normal canonical forms of matrices.

3.2.1 | Case ${p_2}^2 + 4p_1 < 0$

When the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ is isomorphic to the algebra of the complexes $\mathbb{C}$, we have the following proposition.

Proposition 3.1. *If ${p_2}^2 + 4p_1 < 0$, $\gamma = \sqrt{-{p_2}^2 - 4p_1}$, and $\varphi(x, y) = (ax + by, cx + dy)$, then the solutions f and g of the PDE (2) given in Theorem 3.1 are the following:*

$$f(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\cos\left(\frac{\gamma}{2}(cx+dy)\right) - \frac{p_2}{\gamma}\sin\left(\frac{\gamma}{2}(cx+dy)\right)\right), \tag{26}$$

and

$$g(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\frac{(-{p_2}^2-\gamma^2)\sin\left(\frac{\gamma}{2}(cx+dy)\right)}{2p_1\gamma}\right). \tag{27}$$

Proof. If ${p_2}^2 + 4p_1 < 0$, then $p_1 < 0$ and

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} 0 & 2p_1 \\ \gamma & p_2 \end{pmatrix}\begin{pmatrix} \frac{p_2}{2} & -\frac{\gamma}{2} \\ \frac{\gamma}{2} & \frac{p_2}{2} \end{pmatrix}\begin{pmatrix} \frac{-p_2}{2p_1\gamma} & \frac{1}{\gamma} \\ \frac{1}{2p_1} & 0 \end{pmatrix}.$$

Thus, we have

$$e^{R(\varphi(x,y))} = e^{ax+by}e^{\frac{p_2}{2}(cx+dy)}Me^{B(cx+dy)}M^{-1},$$

where R is the first fundamental representation of $\mathbb{A}^1(p_1, p_2)$,

$$e^{B(cx+dy)} = \begin{pmatrix} \cos\left(\frac{\gamma}{2}(cx+dy)\right) & -\sin\left(\frac{\gamma}{2}(cx+dy)\right) \\ \sin\left(\frac{\gamma}{2}(cx+dy)\right) & \cos\left(\frac{\gamma}{2}(cx+dy)\right) \end{pmatrix},$$

and

$$M = \begin{pmatrix} 0 & 2p_1 \\ \gamma & p_2 \end{pmatrix}.$$

Therefore, the expressions for f and g are given by (26) and (27), respectively. □

3.2.2 | Case $p_2{}^2 + 4p_1 = 0$

When the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ is isomorphic to the algebra spanned by the Jordan canonical form, we have the following proposition.

Proposition 3.2. *If $p_2{}^2 + 4p_1 = 0$, and $\varphi(x, y) = (ax + by, cx + dy)$, then the solutions f and g of the PDE (2) given in Theorem 3.1 are the following:*

$$f(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \left(\frac{-p_2(cx + dy) + 2}{2} \right), \tag{28}$$

and

$$g(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}(cx + dy). \tag{29}$$

Proof. If $p_2{}^2 + 4p_1 = 0$, then $p_1 = -p_2{}^2/4$, and

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} -\frac{p_2}{2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{p_2}{2} & 1 \\ 0 & \frac{p_2}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_2}{2} \end{pmatrix}.$$

Thus, we have

$$e^{R(\varphi(x,y))} = e^{ax+by+\frac{p_2}{2}(cx+dy)} \begin{pmatrix} -\frac{p_2}{2} & 1 \\ 1 & 0 \end{pmatrix} e^{(cx+dy)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_2}{2} \end{pmatrix},$$

then

$$e^{R(\varphi(x,y))} = e^{ax+by+\frac{p_2}{2}(cx+dy)} \begin{pmatrix} \frac{-p_2(cx+dy)+2}{2} & \frac{-p_2{}^2(cx+dy)}{4} \\ cx + dy & \frac{p_2(cx+dy)+2}{2} \end{pmatrix}.$$

Therefore, in this case, the expressions for f and g are given by (28) and (29), respectively. □

3.2.3 | Case $p_2{}^2 + 4p_1 > 0$

When the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ is isomorphic to the direct sum of $\mathbb{R}$ and $\mathbb{R}$, we have the following proposition.

Proposition 3.3. *Let $p_2{}^2 + 4p_1 > 0$, $\varphi(x, y) = (ax + by, cx + dy)$, and $\gamma = \sqrt{p_2{}^2 + 4p_1}$. Thus, the solutions f and g of the PDE (2) given in Theorem 3.1 are the following:*

1. *If $p_1 \neq 0$,*

$$f(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma - p_2)e^{\frac{\gamma}{2}(cx+dy)} + (\gamma + p_2)e^{-\frac{\gamma}{2}(cx+dy)}}{2\gamma}, \tag{30}$$

and

$$g(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma^2 - p_2{}^2)e^{\frac{\gamma}{2}(cx+dy)} - (\gamma^2 - p_2{}^2)e^{-\frac{\gamma}{2}(cx+dy)}}{4p_1\gamma}. \tag{31}$$

2. *If* $p_1 = 0$,

$$f(x,y) = e^{ax+by}, \tag{32}$$

and

$$g(x,y) = \frac{1}{p_2} e^{ax+by} \left(-1 + e^{p_2(cx+dy)}\right). \tag{33}$$

Proof. If ${p_2}^2 + 4p_1 > 0$, we have that the matrix

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix},$$

is diagonalizable. If $p_1 \neq 0$, then

$$\begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} 2p_1 & 2p_1 \\ p_2+\gamma & p_2-\gamma \end{pmatrix} \begin{pmatrix} \frac{p_2+\gamma}{2} & 0 \\ 0 & \frac{p_2-\gamma}{2} \end{pmatrix} \begin{pmatrix} \frac{-p_2+\gamma}{4p_1\gamma} & \frac{p_1}{2p_1\gamma} \\ \frac{p_2+\gamma}{4p_1\gamma} & \frac{-p_1}{2p_1\gamma} \end{pmatrix},$$

where $\gamma = \sqrt{{p_2}^2 + 4p_1}$. Therefore, in this case, the expressions for f and g are given by (30) and (31), respectively.

If $p_1 = 0$, then

$$\begin{pmatrix} 0 & 0 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} p_2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} \frac{1}{p_2} & 0 \\ \frac{1}{p_2} & 1 \end{pmatrix}.$$

Therefore, in this case, the expressions for f and g are given by (32) and (33), respectively. □

3.3 | Solutions for the one-dimensional heat equation

In the following example, we construct solutions for the one-dimensional heat equation.

Example 3.2. Now consider the one-dimensional heat Equation (3). In this case, we change the variable y by t, $A = \alpha$, $E = -1$, B, C, D, and F are equal to zero. So,

$$p_1 = -\frac{\alpha a^2 - b}{\alpha c^2}, \qquad p_2 = -\frac{2\alpha ac - d}{\alpha c^2}.$$

Suppose that $\alpha = 1/7$, $a = c = \sqrt{7}$, $b = 1$, and $d = 2$, then

$$p_1 = -\frac{\alpha a^2 - b}{\alpha c^2}, \qquad p_2 = -\frac{2\alpha ac - d}{\alpha c^2}.$$

Thus, $p_1 = 0$ and $p_2 = 0$. By Proposition 3.2

$$f(x,t) = e^{\sqrt{7}x+t}, \qquad g(x,t) = e^{\sqrt{7}x+t}\left(\sqrt{7}x + 2t\right),$$

and by Theorem 3.1 they are solutions. For the same value of α, we may choose values for the constants a, b, c, and d with the only condition that $c \neq 0$, so we can obtain another solutions to the heat equation.

3.4 | Solutions by trigonometric and hyperbolic functions of φ

Now, we consider the trigonometric sine and cosine functions instead of the exponential function.

Proposition 3.4. *Suppose* $D = 0$ *and* $E = 0$ *in the PDE (2), and* $Ac^2 + Bcd + Cd^2 \neq 0$. *Let* p_1 *and* p_2 *be defined by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 - F}{Ac^2 + Bcd + Cd^2}, \tag{34}$$

and

$$p_2 = -\frac{2Aac + B(ad+bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}. \tag{35}$$

Let $\mathcal{T}$ denote any of the trigonometric functions

$$\mathcal{S}(x,y)=\sin(\varphi(x,y)),\qquad \mathcal{C}(x,y)=\cos(\varphi(x,y)),$$

defined with respect to the $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$ product. Thus, the functions f and g given by

$$(f,g)=\mathcal{T}(ax+by,cx+dy), \tag{36}$$

are solutions of the PDE (2) with $D=0$ and $E=0$.

Proof. Proof is similar to that of Theorem 3.1, but in this case, it is used that $\mathcal{T}=-\mathcal{T}'_\varphi$. □

In the following proposition, the hyperbolic sine and cosine functions are now considered instead of the exponential function.

Proposition 3.5. *Suppose $D=0$, $E=0$ in PDE (2), and $Ac^2+Bcd+Cd^2\neq 0$. Let p_1 and p_2 be defined by*

$$p_1=-\frac{Aa^2+Bab+Cb^2+F}{Ac^2+Bcd+Cd^2}, \tag{37}$$

and

$$p_2=-\frac{2Aac+B(ad+bc)+2Cbd}{Ac^2+Bcd+Cd^2}. \tag{38}$$

Let $\mathcal{H}$ denote any of the hyperbolic functions

$$\mathcal{S}(x,y)=\sinh(\varphi(x,y)),\qquad \mathcal{C}(x,y)=\cosh(\varphi(x,y)),$$

defined with respect to the $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$ product. Thus, the functions f and g defined by

$$(f,g)=\mathcal{H}(ax+by,cx+dy), \tag{39}$$

are solutions of the PDE (2) with $D=0$ and $E=0$.

Proof. Proof is similar to that of Theorem 3.1, but in this case it is used that $\mathcal{H}=\mathcal{H}'_\varphi$. □

4 | PDEs $Au_{xx}+Bu_{xy}+Cu_{yy}=0$

In the following theorem, we determine algebras $\mathbb{A}$ with respect to which the components f and g of all the $\varphi\mathbb{A}$-differentiable functions are solutions for the PDE (1).

Theorem 4.1. *Consider the PDE (1) and the affine planar vector field φ given in (5). Suppose that $Ac^2+Bcd+Cd^2\neq 0$. Thus, for the algebra $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$ with*

$$p_1=-\frac{Aa^2+Bab+Cb^2}{Ac^2+Bcd+Cd^2}, \tag{40}$$

and

$$p_2=-\frac{2Aac+B(ad+bc)+2Cbd}{Ac^2+Bcd+Cd^2}, \tag{41}$$

the components f and g of all the $\varphi(\mathbb{A})$-differentiable function are solutions for the PDE (1).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$. The equalities (40) and (41) are equivalent to

$$\begin{aligned} A(a^2 + p_1c^2) + B(ab + p_1cd) + C(b^2 + p_1d^2) = 0, \\ A(2ac + p_2c^2) + B(ad + bc + p_2cd) + C(2bd + p_2d^2) = 0, \end{aligned} \tag{42}$$

respectively. From (15) and (42), it can be obtained

$$A\varphi_x^2 + B\varphi_x\varphi_y + C\varphi_y^2 = 0. \tag{43}$$

From this and the equalities (14), by multiplying $\mathcal{F}'_\varphi$ with respect to $\mathbb{A}$, we get

$$A\mathcal{F}_{xx} + B\mathcal{F}_{xy} + C\mathcal{F}_{yy} = 0. \tag{44}$$

Since $\mathcal{F} = (f, g)$, we have that f and g are solutions for (2). □

Theorem 4.1 is a generalization of a well-known and important result, as we see in the following corollary.

Corollary 4.1. *Suppose that $A = 1$, $B = 0$, $C = 1$, and $\varphi(x, y) = (x, y)$. Then, PDE (1) is the Laplace's equation $u_{xx}+u_{yy} = 0$, and $p_1 = -1$ and $p_2 = 0$. Thus, $\mathbb{A} = \mathbb{A}_1(-1, 0)$ is the algebra of the complex numbers $\mathbb{A} = \mathbb{C}$, the $\varphi\mathbb{A}$-differentiability corresponds to the usual complex differentiability, and the components of the $\varphi\mathbb{A}$-differentiable functions are solutions of the Laplace's equation.*

The following example gives solutions for the Laplace's equation.

Example 4.1. Suppose that $A = 1, B = 0, C = 1$, and $\varphi(x, y) = (x + 2y, 3x + 4y)$. Then, PDE (1) is the Laplace's Equation (4). By Theorem 4.1,

$$p_1 = -\frac{1}{5}, \qquad p_2 = \frac{-22}{25}.$$

Then, for $\mathbb{A} = \mathbb{A}_1(-1/5, -22/25)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (1). Since

$$\begin{aligned} (\varphi(x, y))^2 &= \left((x + 2y)^2 - \frac{1}{5}(3x + 4y)^2, 2(x + 2y)(3x + 4y) - \frac{22}{25}(3x + 4y)^2\right) \\ &= \frac{1}{25}\left(-20x^2 + 20y^2 - 20xy, -48x^2 + 48y^2 - 28xy\right), \end{aligned}$$

we have that

$$f(x, y) = \frac{1}{5}(-4x^2 + 4y^2 - 4xy), \qquad g(x, y) = \frac{1}{25}(-48x^2 + 48y^2 - 28xy),$$

are solutions for the Laplace's equation.

We consider the one-dimensional wave equation in the following example.

Example 4.2. Suppose that $A = 1, B = 0, C = -1$, and $\varphi(x, y) = (x + 2y, 3x + 4y)$. Then, PDE (1) is the classical one-dimensional wave equation given in (4). Thus, by (40) and (41),

$$p_1 = -\frac{3}{7}, \qquad p_2 = \frac{-10}{7}.$$

Let $\mathbb{A} = \mathbb{A}_1(-3/7, -10/7)$. Then, by Theorem 4.1, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for the wave equation. Since

$$\begin{aligned} (\varphi(x, y))^2 &= \left((x + 2y)^2 - \frac{3}{7}(3x + 4y)^2, 2(x + 2y)(3x + 4y) - \frac{10}{7}(3x + 4y)^2\right) \\ &= -\frac{1}{7}(20x^2 + 20y^2 + 4xy, 48x^2 + 48y^2 + 100xy), \end{aligned}$$

we have that

$$f(x, y) = -\frac{20x^2 + 20y^2 + 4xy}{7}, \qquad g(x, y) = -\frac{48x^2 + 48y^2 + 100xy}{7},$$

are solutions for the wave equation.

Suppose that $(k,l)\neq(0,0)$, that is, $\varphi(x,y)=(x+2y+k,3x+4y+l)$. Since

$$\begin{aligned}(\varphi(x,y))^2 &= \left((x+2y+k)^2-\frac{3}{7}(3x+4y+l)^2\right)e_1\\ &\quad+\left(2(x+2y+k)(3x+4y+l)-\frac{10}{7}(3x+4y+l)^2\right)e_2\\ &=\frac{1}{7}(7k^2-3l^2-20x^2+14kx-18lx-20y^2+28ky-24ly-44xy)e_1\\ &\quad+\frac{1}{7}(-10l^2+14kl-48x^2+42kx-46lx-48y^2+56ky-52ly-100xy)e_2,\end{aligned}$$

we have that

$$f(x,y)=\frac{7k^2-3l^2-20x^2+14kx-18lx-20y^2+28ky-24ly-44xy}{7},$$

and

$$g(x,y)=\frac{-10l^2+14kl-48x^2+42kx-46lx-48y^2+56ky-52ly-100xy}{7},$$

are solutions for the wave equation.

In the following example, we consider again the one-dimensional wave equation.

Example 4.3. Consider $A=1$, $B=0$, and $C=-1$ in (4). Let φ be defined by $\varphi(x,y)=(y,x)$. Thus, by (40) and (41), $p_1=1$, and $p_2=0$. Therefore, for $\mathbb{A}=\mathbb{A}_1(1,0)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (1). Since

$$(\varphi(x,y))^2=\left(x^2+y^2,2xy\right),$$

we have that

$$f(x,y)=x^2+y^2,\qquad g(x,y)=2xy,$$

are solutions for the wave equation.

5 | SOLUTIONS OF PDES AND $\varphi\mathbb{A}$-DIFFERENTIABLE FUNCTIONS

The solutions of PDEs (1) are components of $\varphi\mathbb{A}$-differentiable function, as we see in the following theorem.

Theorem 5.1. *Consider a planar vector field φ given in (5) such that $Ac^2+Bcd+Cd^2\neq 0$, and suppose that equalities (40), (41) are satisfied.*

1) If $p_1(ad-bc)\neq 0$, then the $\varphi\mathbb{A}$ Cauchy–Riemann Equation (16) can be expressed by

$$\begin{aligned}g_x&=\frac{-ab+cdp_1-bcp_2}{(ad-bc)p_1}f_x+\frac{a^2-c^2p_1+acp_2}{(ad-bc)p_1}f_y,\\ g_y&=\frac{-b^2+d^2p_1-bdp_2}{(ad-bc)p_1}f_x+\frac{ab-cdp_1+adp_2}{(ad-bc)p_1}f_y,\end{aligned}\tag{45}$$

from which we obtain

$$(g_x)_y-(g_y)_x=\frac{(ad-bc)}{p_1}(Af_{xx}+Bf_{xy}+Cf_{yy}).\tag{46}$$

Therefore, if f is a solution of PDE (1) and

$$g=\int g_x dx+\int\left[g_y-\frac{\partial}{\partial y}\int g_x dx\right]dy,\tag{47}$$

then $\mathcal{F}=(f,g)$ is $\varphi\mathbb{A}$-differentiable.

2) If $(ad-bc)\neq 0$, then the $\varphi\mathbb{A}$ Cauchy–Riemann Equation (16) can be expressed by

$$\begin{aligned}f_x&=\frac{-ab+cdp_1-adp_2}{ad-bc}g_x+\frac{a^2-c^2p_1+acp_2}{ad-bc}g_y,\\ f_y&=\frac{-b^2+d^2p_1-bdp_2}{ad-bc}g_x+\frac{ab-cdp_1+bcp_2}{ad-bc}g_y,\end{aligned}\tag{48}$$

from which we obtain

$$(f_x)_y - (f_y)_x = (ad - bc)(Ag_{xx} + Bg_{xy} + Cg_{yy}). \tag{49}$$

Therefore, if g is a solution of PDE (1), and

$$f = \int f_y dy + \int \left[f_x - \frac{\partial}{\partial x} \int f_y dy \right] dx, \tag{50}$$

then $\mathcal{F} = (f, g)$ is $\varphi\mathbb{A}$-differentiable.

Proof. The systems (45) and (48) can be obtained from system (16). In the first case, we have

$$p_1(ad - bc)((g_x)_y - (g_y)_x) = (ad - bc)^2(Af_{xx} + Bf_{xy} + Cf_{yy});$$

thus, we obtain (46). In the second case, we have

$$(ad - bc)((f_x)_y - (f_y)_x) = (ad - bc)^2(Ag_{xx} + Bg_{xy} + Cg_{yy});$$

thus, we obtain (49). Since $g_{yx} = g_{xy}$ if f is a solution of (1), we have that there exists a function $g(x, y)$ (uniquely defined under a constant additive) which satisfies (45). In the same way, since $f_{yx} = f_{xy}$ if g is a solution of (1), we have that there exists a function $f(x, y)$ (uniquely defined under a constant additive) which satisfies (48). The function $\mathcal{F} = (f, g)$ so constructed satisfies the corresponding $\varphi\mathbb{A}$ Cauchy–Riemann equations. Since φ is a linear isomorphism, hypotheses of Theorem 1.2 of preprint arXiv 1805.10524 are satisfied. Therefore, $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable. □

In the following example we give a solution of the one-dimensional wave equation and see that this is the first component of a pre-twisted differentiable function.

Example 5.1. Consider $\mathbb{A} = \mathbb{A}_1(1, 0)$, $\varphi(x, y) = (y, x)$, and one-dimensional wave equation. By Example 4.3, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of this equation. By Theorem 5.1, these are all the solutions. The $\varphi\mathbb{A}$ Cauchy–Riemann Equation (16) is defined by

$$f_x = g_y, \qquad f_y = g_x. \tag{51}$$

The function $f(x, y) = x^3 + 3xy^2$ is a solution of the one-dimensional wave equation. Then, by 1) of Theorem 5.1, we have

$$g(x, y) = 3x^2 y + y^3 + k,$$

where k is a constant. In this case, $\mathcal{F} = (f, g)$ is given by $\mathcal{F}(x, y) = (\varphi(x, y))^3 + (0, k)$.

6 | 3RD-ORDER PDES

Now, consider the 3rd-order PDEs

$$Gu_{xxx} + Hu_{xxy} + Ku_{xyy} + Lu_{yyy} + Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0. \tag{52}$$

We are looking for solutions defined by the components of the exponential functions (6). For affine planar vector filed φ the third-order partial derivatives of the $\varphi\mathbb{A}$-differentiable functions $\mathcal{F} = (f, g)$ are given by

$$\mathcal{F}_{xxx} = \mathcal{F}'''_\varphi \varphi_x^3, \;\; \mathcal{F}_{xxy} = \mathcal{F}v_\varphi \varphi_x^2 \varphi_y, \;\; \mathcal{F}_{xyy} = \mathcal{F}'''_\varphi \varphi_x \varphi_y^2, \;\; \mathcal{F}_{yyy} = \mathcal{F}'''_\varphi \varphi_y^3. \tag{53}$$

From the product with respect to $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ and the proposed form of φ in (5), we have

$$\begin{aligned}\varphi_x^3 &= (a^3, 3a^2c + c^3p_1) + (3ac^2 + c^3p_2)(p_1, p_2),\\ \varphi_x^2\varphi_y &= (a^2b, 2abc + a^2d + c^2dp_1) + (bc^2 + 2acd + c^2dp_2)(p_1, p_2),\\ \varphi_x\varphi_y^2 &= (ab^2, 2abd + b^2c + cd^2p_1) + (ad^2 + 2bcd + c^2dp_2)(p_1, p_2),\\ \varphi_y^3 &= (b^3, 3b^2d + d^3p_1) + (3bd^2 + d^3p_2)(p_1, p_2).\end{aligned} \tag{54}$$

For 3rd-order PDEs, we have the following theorem.

Theorem 6.1. *Consider the PDE (52), the affine planar vector field φ given in (5), and the quadratic system of equations*

$$\begin{aligned}&(Gc^3 + Hc^2d + Kcd^2 + Ld^3)xy\\ &+(3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2)x\\ &+Ga^3 + Ha^2b + Kab^2 + Lb^3 + Aa^2 + Bab + Cb^2 + Da + Eb + F = 0,\\ &(Gc^3 + Hc^2d + Kcd^2 + Ld^3)y^2 + (Gc^3 + H(c^2d + bc^2) + Kcd^2 + Ld^3)x\\ &+(3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2)y + 3Ga^2c\\ &+2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d + 2Aac + B(ad + bc) + 2Cbd + Dc + Ed = 0.\end{aligned} \tag{55}$$

If (p_1, p_2) is a solution of the quadratic system (55), for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, the component functions f and g of the exponential function (6) defined with respect to $\mathbb{A}$ are solutions of the PDE (52).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 are a solution of the quadratic system (55). Using the equalities (12), (14), (15), (53), (54), and the obtained by substituting p_1, p_2 in (55), we obtain that columns of $\mathcal{E}$ are solutions for (52). □

The following corollary could help us to construct solutions for (52).

Corollary 6.1. *If φ is satisfies*

$$c^2(Gc + Hd) + d^2(Kc + Ld) = 0, \tag{56}$$

and

$$3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2 \neq 0, \tag{57}$$

then the quadratic system (55) given in Theorem 6.1 reduces to a the linear system with solutions

$$p_1 = -\frac{Ga^3 + Ha^2b + Kab^2 + Lb^3 + Aa^2 + Bab + Cb^2 + Da + Eb + F}{3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2}, \tag{58}$$

and

$$p_2 = -\frac{3Ga^2c + 2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d + 2Aac + B(ad + bc) + 2Cbd + Dc + Ed}{3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2 + Ac^2 + Bcd + Cd^2} - Hbc^2p_1. \tag{59}$$

Proof. Proof follows from Theorem 6.1 since

$$c^2(Gc + Hd) + d^2(Kc + Ld) = Gc^3 + Hc^2d + Kcd^2 + Ld^3.$$

□

In the following example we see that the exponential function of φ can be used to construct solutions for third-order PDEs.

Example 6.1. Consider the PDE

$$u_{xxx} + 2u_{xxy} + 2u_{xyy} + 4u_{yyy} + 5u_{xx} + 6u_{xy} + 7u_{yy} + 8u_x + 9u_y + 10u = 0. \tag{60}$$

For $c = 2$ and $d = -1$, we have $c + 2d = 0$, $2c + 4d = 0$, then from Corollary 6 is satisfied. Thus,

$$p_1 = -\frac{a^3 + 2a^2b + 2ab^2 + 4b^3 + 5a^2 + 6ab + 7b^2 + 8a + 9b + 10}{12a + 2(4b - 4a) + 2(a - 4b) + 12b + 20 - 12 + 7},$$

and

$$p_2 = -\frac{6a^2 + 4(2ab - a^2) + 2(2b^2 - 2ab) - 12b^2 + 20a + 6(-a + 2b) - 14b + 16 - 9}{12a + 2(4b - 4a) + 2(a - 4b) + 12b + 20 - 12 + 7} + 8p_1.$$

If we take $a = b = 0$, then $\alpha = -2/3$ and $\beta = -87/5$. In this case, $p_2^2 + 4p_1 > 0$, so functions f and g given in Proposition 3.3 are solutions for the PDE (60).

Consider the 3rd-order PDE

$$Gu_{xxx} + Hu_{xxy} + Ku_{xyy} + Lu_{yyy} = 0. \tag{61}$$

We have the following theorem for 3rd PDEs of the form (61).

Theorem 6.2. *Consider the PDE (61), the affine planar vector field φ given in (5), and the quadratic system of equations*

$$\begin{aligned}
&(Gc^3 + Hc^2d + Kcd^2 + Ld^3)xy \\
&+(3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)x \\
&+Ga^3 + Ha^2b + Kab^2 + Lb^3 = 0, \\
\\
&(Gc^3 + Hc^2d + Kcd^2 + Ld^3)y^2 + (Gc^3 + H(c^2d + bc^2) + Kcd^2 + Ld^3)x \\
&+(3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)y + 3Ga^2c \\
&+2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d = 0.
\end{aligned} \tag{62}$$

If (p_1, p_2) is a solution of the quadratic system (62), for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, the components of all the $\varphi\mathbb{A}$-differentiable function are solutions of the PDE (52).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 are a solution of the quadratic system (62). Using the equalities (14), (15), (53), (54), and the obtained by substituting p_1, p_2 in (62), we obtain that columns of the $\varphi\mathbb{A}$-differentiable functions are solutions for (52). □

7 | 4TH-ORDER PDES

The *bi-harmonic equation* is the 4th-order PDE

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0, \tag{63}$$

the *bi-wave equation* is the 4th-order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} = 0, \tag{64}$$

and the *bi-telegraph equation* is the 4th-order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} - \lambda^4 u = 0; \tag{65}$$

see Pogorui et al.[8] For φ being an affine planar vector field, the fourth-order partial derivatives $\mathcal{F}_{xxxx}$, $\mathcal{F}_{xxyy}$, and $\mathcal{F}_{yyyy}$ of $\varphi\mathbb{A}$-differentiable functions $\mathcal{F}$ are given by

$$\mathcal{F}_{xxxx} = \mathcal{F}_\varphi'''' \varphi_x^4, \qquad \mathcal{F}_{xxyy} = \mathcal{F}_\varphi'''' \varphi_x^2\varphi_y^2, \qquad \mathcal{F}_{yyyy} = \mathcal{F}_\varphi'''' \varphi_y^4. \tag{66}$$

From the $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ product and the proposed form of φ in (5), we have

$$\begin{aligned}
\varphi_x^4 &= c^4(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + 4ac^3(p_1p_2, p_2^2 + p_1) + 6a^2c^2(p_1, p_2) + a^3(a, 4c),\\
\varphi_x^2\varphi_y^2 &= c^2d^2(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + (2cd(ad + bc)p_2 + a^2d^2 + 4abcd + b^2c^2)(p_1, p_2)\\
&\quad + (a^2b^2, 2(ad + bc)(cdp_1 + ab)),\\
\varphi_y^4 &= d^4(p_1p_2^2 + p_1^2, p_2^3 + 2p_1p_2) + 4bd^3(p_1p_2, p_2^2 + p_1) + 6b^2d^2(p_1, p_2) + b^3(b, 4d).
\end{aligned} \tag{67}$$

We have the following theorem for the bi-harmonic equation.

Theorem 7.1. *Consider the PDE (63), the affine planar vector field φ given in (5), and the cubic system of two equations*

$$\begin{aligned}
&(c^2 + d^2)^2(xy^2 + x^2) + 4(ac + bd)(c^2 + d^2)xy\\
&+ 2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)x + (a^2 + b^2)^2 = 0,\\
&(c^2 + d^2)^2(y^3 + 2xy) + 4(ac + bd)(c^2 + d^2)(y^2 + x)\\
&+ 2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)y + 4(a^2 + b^2)(ac + bd) = 0.
\end{aligned} \tag{68}$$

If (p_1, p_2) is a solution of the cubic system (68), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the PDE (63).

Proof. Let $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, where p_1 and p_2 are a solution of the cubic system (68). Using the equalities (66), (67), and (68), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for the bi-harmonic equation (63). □

In the following example we use the multiplicative inverse of φ to construct bi-harmonic functions.

Example 7.1. If $\varphi(x, y) = (x + y + k, x - y + l)$, then $p_1 = -1$ and $p_2 = 0$ satisfy conditions (68). Thus, for $\mathbb{A} = \mathbb{C}$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-harmonic Equation (63). Function

$$(x + y + k, x - y + l)^{-1} = \left(\frac{x + y + k}{(x + y + k)^2 + (x - y + l)^2}, \frac{x - y + l}{(x + y + k)^2 + (x - y + l)^2} \right),$$

has components

$$f(x, y) = \frac{x + y + k}{(x + y + k)^2 + (x - y + l)^2}, \qquad g(x, y) = \frac{x - y + l}{(x + y + k)^2 + (x - y + l)^2},$$

which are bi-harmonic functions. Also function $(x+y+k, x-y+l)^4$ has components

$$\begin{aligned}
f(x, y) &= k^4 + l^4 - 6k^2l^2 - 4x^4 - 8kx^3 - 8lx^3 - 24klx^2 + 4k^3x + 4l^3x - 12kl^2x - 12k^2lx\\
&\quad - 4y^4 - 8ky^3 + 8ly^3 + 24kly^2 + 24x^2y^2 + 24kxy^2 + 24lxy^2 + 4k^3y - 4l^3y\\
&\quad - 12kl^2y + 12k^2ly + 24kx^2y - 24lx^2y + 24k^2xy - 24l^2xy,\\
g(x, y) &= -4kl^3 + 4k^3l + 8kx^3 - 8lx^3 + 12k^2x^2 - 12l^2x^2 + 4k^3x - 4l^3x - 12kl^2x + 12k^2lx\\
&\quad - 8ky^3 - 8ly^3 - 16xy^3 - 12k^2y^2 + 12l^2y^2 - 24kxy^2 + 24lxy^2 - 4k^3y - 4l^3y\\
&\quad + 12kl^2y + 12k^2ly + 16x^3y + 24kx^2y + 24lx^2y + 48klxy,
\end{aligned}$$

which are bi-harmonic functions.

In the following example we give a planar vector field φ, which is not the identity function, for which the algebra of complex numbers can be used to construct bi-harmonic functions.

Example 7.2. If $\varphi(x, y) = (y + k, -x + l)$, then $p_1 = -1$ and $p_2 = 0$ satisfy conditions (68). Thus, for $\mathbb{A} = \mathbb{C}$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-harmonic Equation (63).

We have the following theorem for the bi-wave equation.

Theorem 7.2. *Consider the PDE (64), the affine planar vector field φ given in (5), and the cubic system of two equations*

$$\begin{aligned}&(c^2-d^2)^2(xy^2+x^2)+4(ac-bd)(c^2-d^2)xy\\&+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)x+(a^2-b^2)^2=0,\\&(c^2-d^2)^2(y^3+2xy)+4(ac-bd)(c^2-d^2)(y^2+x)\\&+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)y+4(a^2-b^2)(ac-bd)=0.\end{aligned} \tag{69}$$

If (p_1,p_2) is a solution of the cubic system (69), then for $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-wave Equation (64).

Proof. Let $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$, where p_1 and p_2 are a solution of the cubic system (69). Using the equalities (66), (67), and (69), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (64). □

In the following example, the multiplicative inverse and the fourth power function of φ are used to construct solutions for the bi-wave equation.

Example 7.3. If $\varphi(x,y)=(y+k,x-y+l)$, then $p_1=-1/4$ and $p_2=1$ satisfy conditions (69). Thus, for $\mathbb{A}=\mathbb{A}_1^2(1/4,1)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-wave Equation (64). Function φ^{-1} is $\varphi\mathbb{A}$-differentiable, and

$$(y+k,x-y+l)^{-1}=(f(x,y),g(x,y)),$$

where

$$\begin{aligned}f(x,y)&=\frac{4x+4k+4l}{(x+y)^2+(4k+2l)(x+y)+(2k+l)^2},\\g(x,y)&=\frac{-4x+4y-4l}{(x+y)^2+(4k+2l)(x+y)+(2k+l)^2},\end{aligned}$$

which are solutions of the bi-wave equation. In the same way for φ^4, we have

$$(y+k,x-y+l)^4=(f(x,y),g(x,y)),$$

where

$$\begin{aligned}f(x,y)=&\frac{-3x^4-16kx^3-12lx^3-24k^2x^2-18l^2x^2-48klx^2-12l^3x-48kl^2x-48k^2lx}{16}\\&\frac{+5y^4+32ky^3+12ly^3+12xy^3+72k^2y^2+6l^2y^2+48kly^2+6x^2y^2+48kxy^2}{16}\\&\frac{+12lxy^2+64k^3y-4l^3y+48k^2ly-4x^3y-12lx^2y+48k^2xy-12l^2xy}{16}\\&\frac{+16k^4-3l^4-16kl^3-24k^2l^2}{16},\\g(x,y)=&\frac{x^4+6kx^3+4lx^3+12k^2x^2+6l^2x^2+18klx^2+18kl^2x+24k^2lx-y^4-6ky^3}{2}\\&\frac{-2ly^3-2xy^3+8k^3x+4l^3x-12k^2y^2-6kly^2-6kxy^2-8k^3y}{2}\\&\frac{+2l^3y+6kl^2y+2x^3y+6kx^2y+6lx^2y+6l^2xy+12klxy}{2}\\&\frac{+l^4+6kl^3+12k^2l^2+8k^3l}{2},\end{aligned}$$

which are solutions of the bi-wave equation.

In the following example, the multiplicative inverse function of φ is used to construct solutions for the bi-wave equation.

Example 7.4. If $\varphi(x,y)=(x+y+k,x+l)$, then $p_1=0$ and $p_2=-2$ satisfy conditions (69). Thus, for $\mathbb{A}=\mathbb{A}_1^2(0,-2)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-wave Equation (64). Function φ^{-1} is $\varphi\mathbb{A}$-differentiable, and

$$(x+y+k,x+l)^{-1}=(f(x,y),g(x,y)),$$

where

$$f(x,y)=\frac{1}{x+y+k},\qquad g(x,y)=\frac{-x-l}{-x^2+y^2-2lx+2(k-l)y+k^2-2kl},$$

which are solutions of the bi-wave equation.

The exponential, trigonometric, and hyperbolic functions of φ have components which are solutions for the bi-telegraph equation, as we see in the following theorem.

Theorem 7.3. *Consider the PDE (65), the affine planar vector field φ given in (5), and the cubic system of two equations*

$$\begin{aligned}&(c^2-d^2)^2(xy^2+x^2)+4(ac-bd)(c^2-d^2)xy\\&+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)x+(a^2-b^2)^2-\lambda^4=0,\\&(c^2-d^2)^2(y^3+2xy)+4(ac-bd)(c^2-d^2)(y^2+x)\\&+2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)y+4(a^2-b^2)(ac-bd)=0.\end{aligned}\tag{70}$$

If (p_1,p_2) is a solution of the cubic system (70), then for $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$, the components of the functions

$$e^{\varphi(x,y)},\ \sin(\varphi(x,y)),\ \cos(\varphi(x,y)),\ \sinh(\varphi(x,y)),\ \cosh(\varphi(x,y)),$$

are solutions of the bi-telegraph equation (65).

Proof. Let $\mathbb{A}=\mathbb{A}_1(p_1,p_2)$, where p_1 and p_2 are a solution of the cubic system (70). Using the equalities (66), (67), and (70), we obtain that components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (65). □

In the following example, the exponential function of φ is used to construct solutions for the bi-telegraph equation.

Example 7.5. If $\varphi(x,y)=(\lambda x+k,\lambda x+\lambda y+l)$, then $p_1=0$ and $p_2=-1$ satisfy conditions (70). Thus, for $\mathbb{A}=\mathbb{A}_1^2(0,-2)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-telegraph equation (65). By Proposition 3.3, components of function $e^{\varphi(x,y)}$ are given by

$$f(x,y)=e^{\lambda x},\qquad g(x,y)=e^{\lambda x}-e^{-\lambda y},$$

which by Theorem 7.3 are solutions of the bi-telegraph equation.

In the following example, the exponential and sine functions of φ are used to construct solutions for the bi-telegraph equation.

Example 7.6. If $\varphi(x,y)=(x-y+k,x+y+l)$, then $p_1=(\lambda/2)^4$ and $p_2=0$ satisfy conditions (70). By Proposition 3.3, for $\mathbb{A}=\mathbb{A}_1^2((\lambda/2)^4,0)$, the components of function $e^{\varphi(x,y)}$ are given by

$$f(x,y)=\frac{e^{x-y}}{2}\left(e^{\frac{\lambda^2}{4}(x+y)}+e^{-\frac{\lambda^2}{4}(x+y)}\right),\qquad g(x,y)=\frac{2e^{x-y}}{\lambda^2}\left(e^{\frac{\lambda^2}{4}(x+y)}-e^{-\frac{\lambda^2}{4}(x+y)}\right),$$

which by Theorem 7.3 are solutions of the bi-telegraph equation. Components of $\sin(\varphi(x,y))$ are given by

$$f(x,y)=\sin(x-y+k)\cos\left(\frac{\lambda^2}{4}(x+y+k)\right),\ g(x,y)=\frac{4}{\lambda^2}\cos(x-y+k)\sin\left(\frac{\lambda^2}{4}(x+y+k)\right).$$

So, by proof of Theorem 7.3, they are solutions of the bi-telegraph equation.

In the following example, the sine and cosine functions of φ are used to construct solutions for the bi-telegraph equation.

Example 7.7. If $\varphi(x,y)=\left(0,\left(\sqrt{\lambda^2+d^2}\right)x+dy+l\right)$, then $p_1=1$ and $p_2=0$ satisfy conditions (70). Thus, for $\mathbb{A}=\mathbb{A}_1^2(0,-2)$, the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-telegraph equation (65). Following proof of Proposition 3.3, we obtain that components of functions $\sin(\varphi(x,y))$ are given by

$$f(x,y)=0,\qquad g(x,y)=\sin\left(\left(\sqrt{\lambda^2+d^2}\right)x+dy+l\right),$$

which by Theorem 7.3 are solutions of the bi-telegraph equation. In the same way, components of functions $\cos(\varphi(x, y))$ are given by

$$f(x, y) = \cos\left(\left(\sqrt{\lambda^2 + d^2}\right)x + dy + l\right), \qquad g(x, y) = 0,$$

and they are solutions of the bi-telegraph equation.

8 | THE CASE $Du_x + Eu_y + Fu = 0$

Now, consider the PDEs

$$Du_x + Eu_y + Fu = 0. \tag{71}$$

For this linear case, the following result gives conditions under which the components of the exponential function (6) are solutions for (71). This is satisfied for any two-dimensional algebra.

Proposition 8.1. *Let (a, b) be a solution of $Dx + Ey + F = 0$, (c, d) a solution of $Dx + Ey = 0$, $\mathbb{A}$ an algebra, and φ defined by (5). Thus, the components of the exponential function (6) defined with respect to $\mathbb{A}$ are solutions for (71).*

Proof. If (a, b) is a solution of $Dx + Ey + F = 0$ and (c, d) of $Dx + Ey = 0$, then $D(a, c) + E(b, d) = -F(1, 0)$. Thus,

$$D\varphi_x e^{\varphi(x,y)} + E\varphi_y e^{\varphi(x,y)} = -Fe^{\varphi(x,y)}.$$

That is, if $(f, g) = e^{\varphi}$, then

$$D(f, g)_x + E(f, g)_y = -F(f, g).$$

Therefore, f and g are solutions for (71). □

In the following example we use the expressions for the exponential function of φ given in Section 3 to construct solutions for the case of first-order PDEs.

Example 8.1. Consider the PDE (71) with $D = 1$, $E = 2$, and $F = 3$. Then $(-1, -1)$ is a solution of $x + 2y = -3$, and $(2, -1)$ is a solution of $x + 2y = 0$. We take $\varphi(x, y) = (-x - y, 2x - y)$. Let $\mathbb{A}$ be the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$, and $\mathcal{E}$ the exponential (6) defined with respect to $\mathbb{A}$.

If ${p_2}^2 + 4p_1 < 0$, and $\gamma = \sqrt{-{p_2}^2 - 4p_1}$, then by Proposition 3.1 the component f and g of $\mathcal{E}$ are given by (26) and (47), which in this case take the form

$$f(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}\left(\cos\left(\frac{\gamma}{2}(2x - y)\right) - \frac{p_2}{\gamma}\sin\left(\frac{\gamma}{2}(2x - y)\right)\right),$$
$$g(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}\left(\frac{(-{p_2}^2 - \gamma^2)\sin\left(\frac{\gamma}{2}(2x - y)\right)}{2p_1\gamma}\right).$$

If ${p_2}^2 + 4p_1 = 0$, then by Proposition 3.2, the components f and g of $\mathcal{E}$ are given by (28) and (29), which in this case take the form

$$f(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}\left(\frac{-p_2(2x - y) + 2}{2}\right),$$
$$g(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}(2x - y).$$

If ${p_2}^2 + 4p_1 > 0$, and $\gamma = \sqrt{{p_2}^2 + 4p_1}$, then by Proposition 3.3, the components f and g of $\mathcal{E}$ are given by

1) (30) and (31) for $p_1 \neq 0$, which in this case take the form

$$f(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}\frac{(\gamma - p_2)e^{\frac{\gamma}{2}(2x-y)} + (\gamma + p_2)e^{-\frac{\gamma}{2}(2x-y)}}{2\gamma},$$
$$g(x, y) = e^{-x-y+\frac{p_2}{2}(2x-y)}\frac{(\gamma^2 - {p_2}^2)e^{\frac{\gamma}{2}(2x-y)} - (\gamma^2 - {p_2}^2)e^{-\frac{\gamma}{2}(2x-y)}}{4p_1\gamma}.$$

2) (32) and (33) for $p_1 = 0$, which in this case take the form

$$f(x, y) = e^{-x-y},$$
$$g(x, y) = \frac{1}{p_2} e^{-x-y} \left(-1 + e^{p_2(2x-y)}\right).$$

9 | CONCLUSIONS

The main contribution of this paper is that it shows that pre-twisted differentiability (see preprint arXiv 1805.10524) is a generalization of complex analysis; it generalizes the result relating harmonic functions to the components of complex analytic functions. In this paper, for each PDE of type (1), and for φ given by (5), by Theorem 4.1, there exists an algebra $\mathbb{A}$ such that all components of $\varphi\mathbb{A}$-differentiable functions are solutions of the given equation. By Theorem 5.1, one has that these are all the solutions. In this way, all these PDEs are solved. In addition, the pre-twisted differentiability can be used to construct solutions for PDEs of the form (2) and for higher order PDEs. In these developments, the pre-twisted differentiability, its generalized Cauchy–Riemann equations, and the algebras $\mathbb{A}_1(p_1, p_2)$ have played a central role.

The method applied in this paper is a more explicit way of the method proposed in Ketchum[1] for solving PDEs of mathematical physics. Several authors cite this work; they put conditions of the type that there exists an algebra $\mathbb{A}$ for which $Ae_1^2 + Be_1e_2 + Ce_2^2 = 0$ is satisfied in order to construct solutions of PDEs of the type (1); see Pogorui et al[8] and Plaksa.[9] In this paper, we require that

$$A\varphi(e_1)^2 + B\varphi(e_1)\varphi(e_2) + C\varphi(e_2)^2 + D\varphi(e_1) + E\varphi(e_2) + Fe_1 = 0,$$

for constructing solutions for PDEs of the type (2). This allows us to build solutions for a wider class of EDPs.

The constructions given in this paper can be generalized to differential equations with a larger number of independent variables. For the case of the 3D Laplace's equation, examples can be given in which the components of the $\varphi\mathbb{A}$-differentiable functions are harmonic functions, but it is not possible to find a 3D linear vector field φ and an algebra $\mathbb{A}$ such that the components of all the φA-differentiable functions are all the harmonic functions of three variables. For the case of homogeneous quadratic polynomials, one can find three examples of families of $\varphi\mathbb{A}$-differentiable functions such that all harmonic quadratic polynomials that are homogeneous can be expressed as linear combination of components of these three families of $\varphi\mathbb{A}$-differentiable functions.

In addition to the applications of the components of pre-twisted differentiability function for solving classical PDEs of the mathematical physics, applications to Elasticity are possible; see Lu[10] and preprint arXiv: 1601.01626. More functional analysis techniques, complex analysis, and matrix theory are applied in Aguirre-Hernández et al[11] and references therein.

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CONFLICT OF INTEREST

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