applied sciences

MDPI

Article

Reliability by Using Weibull Distribution Based on Vibration Fatigue Damage

Jesús M. Barraza-Contreras, Manuel R. Piña-Monarrez * and Roberto C. Torres-Villaseñor

Industrial and Manufacturing Department, Engineering and Technological Institute, Universidad Autónoma de Ciudad Juárez, Ciudad Juárez 32310, Chihuahua, Mexico; al187061@alumnos.uacj.mx (J.M.B.-C.); al153286@alumnos.uacj.mx (R.C.T.-V.)

* Correspondence: manuel.pina@uacj.mx; Tel.: +52-656-330-1229

Abstract: In this paper, a Weibull probabilistic methodology is proposed with an approach to model vibration fatigue damage accumulation using two parameters: Weibull distribution and a nonlinear fatigue damage accumulation model. The damage is cumulated based on the application of a vibration stress profile and is used to determine both the Weibull β and η parameters, and the corresponding component reliability R(t). The vibration fatigue damage is analyzed to accumulate the damage as a stress function for a fatigue life exponent derived with the assistance of the acceleration's force response. The steps to determine the Weibull β and η parameters are estimated based only on the principal vibration stresses σ_1 and σ_2 that allow the reproduction of the vibration fatigue damage. The method's efficiency is based on the probabilistic approach by using the vibration fatigue damage as the Y_i vector that covers the arithmetic mean as well as the β parameter. Finally, the procedure proposed is applied in a practical case where a mechanical component is used as a support for telecommunication connections and is submitted to vibration stress. The results show that using the damage accumulated as the Y_i vector to estimate the parameters allows for the analysis of dynamic and individual applications.

Keywords: random vibration; mechanical fatigue; damage; Weibull distribution; reliability

check for updates

Citation: Barraza-Contreras, J.M.; Piña-Monarrez, M.R.; Torres-Villaseñor, R.C. Reliability by Using Weibull Distribution Based on Vibration Fatigue Damage. *Appl. Sci.* **2023**, *13*, 10291. https://doi.org/10.3390/app131810291

Academic Editors: Stefano Marchesiello and Dario Anastasio

Received: 7 August 2023
Revised: 8 September 2023
Accepted: 11 September 2023
Published: 14 September 2023

1. Introduction

Mechanical components and structures subjected to vibration are affected by dynamic loads, which induce fatigue damage due to the cycling loading application [1,2]. The generated fatigue damage is primarily related to vibration history loading, geometry, and material properties [3]. The generated fatigue damage directly determines the component's reliability, replacement policy, and warranty costs. Thus, the reliability index characteristic has an important role that is determined mathematically and is used to describe the fatigue damage behavior of mechanical components [4]. That description can be performed by using a probabilistic approach [5]. Then, for mechanical components or systems, according to their application and data complexity, an accurate method must be selected to determine an effective level of service life reliability [6]. Now, since data from fatigue damage are affected by significant scatter, and damage is provoked as a response to random forces, gathering fatigue damage data is generally a difficult activity [7]. One of the more commonly applied methods to work with fatigue damage is the Miner's rule [8,9]. To consider the random nature of the generated damage, here, we use a probabilistic time-dependent approach [7]. In the fatigue damage accumulation models, the principal factors involved are load sequence, type of load, overloads, plasticization, and type of material. Consequently, for the damage accumulation analysis, we require a probabilistic concept, or a physical quantity related to the probability of occurrence [10]. Thus, the measurement of damage helps us to calculate the probabilities of failure. Since the accumulation of random vibration fatigue damage entails increasing deterioration, an increasing hazard function is required, and the most recommendable cumulative distribution function (cdf) used to estimate the fatigue damage is the Weibull cdf [11–13]. In a vibration profile, each row has its own stationary frequency and amplitude which allow for the performance of a vibration analysis with a statistical approach. Since it is possible to estimate the stress from the acceleration responses, here, the Weibull distribution is used. Thus, based on the increasing behavior of the cumulated damage, in this paper, the increasing random damage behavior is used in the Weibull cdf to determine the failure percentile that the observed cumulated damage represents in the used vibration profile. Then, once the damage percentiles are determined based on their corresponding cycles of the S–N curve, the Weibull scale parameter is determined. Similarly, the Weibull shape parameter is determined directly from the cumulated response stress of the used profile (see Equations (9) and (10) and Section 3.2). Thus, in Section 2.3, a probabilistic methodology to characterize the fatigue damage induced by random vibration is developed by using the Weibull distribution, which uses a relation of the scale and shape parameters with the mechanical vibration fatigue damage. The methodology includes probabilistic estimations based on the Weibull scale and shape parameters that are governed by a new way of analyzing the fatigue damage accumulation presented in Section 2.1. Then, the vibration analysis is based on the fatigue damage, which is determined directly from the principal stresses σ_1 and σ_2; therefore, the methodology is efficient because it uses the damage as the platform to project and represent its random behavior and consequently the component's fatigue life. With the purpose to assess it, the methodology is applied in Section 3 to a probabilistic failure analysis of a panel support made of cold drawn steel AISI 1025. The mechanical component is submitted to a random vibration loading profile of 10–55 Hertz with an amplitude of 1.5 mm peak to peak during a 2 h period per block. The testing was performed by using an electrodynamic vibration system in which the vibration profile loading was applied 29 times. The component's physical damage results are shown in Section 3.1. The purpose of the method proposed lies in the use of the accumulated fatigue damage D_i instead of the median rank operation. This is illustrated by Equations (12) and (13) which allow the use of the resultant vector Y in the estimation of Weibull parameters that completely reproduces the principal vibration stress values.

The paper is organized as follows. Section 2 includes the generalities of the vibration fatigue damage accumulation and the proposed Weibull fatigue damage analysis method. In Section 3, a numerical application is presented. Section 4 is related to the median rank method comparison. Finally, in Section 5, the conclusions are given.

2. Fatigue Damage

2.1. Fatigue Damage Accumulation

Fatigue damage can be described as a failure mechanism that is manifested when a material tends to fail or break under repeated deflections [14–16]. Thus, a nonlinear model to accumulate the random vibration fatigue damage has been proposed [17] with the purpose of evaluating the fatigue damage of different dynamic loads in mechanical components and structural elements. The acceleration response of the analyzed vibration system is determined by stress as in Equation (1). The applied vibration cycles are determined by the rainflow method [18]. From the S–N material's curve, the corresponding life cycles are determined by using the Basquin Equation [19] as is in Equation (4).

$$\sigma(vib)_i = \sigma_{dynamic} \times A_{res} \tag{1}$$

$$\sigma_{dynamic} = \left(\frac{K m_e \hat{L} C}{I}\right) A \tag{2}$$

$$A_{res} = \frac{2\pi^2 F^2 D_2}{G} \tag{3}$$

$$N_i \times \sigma(vib)_i^b = a^b \tag{4}$$

In Equation (1), $\sigma_{dynamic}$ and A_{res} are the dynamic load factor and the acceleration response, respectively. In Equation (2) [20], K is the stress concentration factor in the mechanical component, m_e is the effective mass, C is the distance to the neutral axis, $\hat{L}$ is the distance from the fixed point of the component to the point of application load, A is the constant of gravity, and I is the moment of inertia. In Equation (3), F is the frequency applied by the vibration power spectral density (PSD) and G is the gravity constant. In Equation (4), N_i is the maximum number of cycles that the material's component can sustain at a vibration load with stress amplitude $\sigma(vib)_i$ and the parameters a and b are constant variables that represent the intercept and the slope of the S–N curve, respectively.

As shown by Equations (1)–(4), because a vibration has a nonlinear behavior [21], the generated fatigue damage also presents a nonlinear behavior. Consequently, the fatigue damage is determined by using Equation (5), where the damage is described by a curve that represents the effect under two-level loading conditions [22], where n_i represents the applied vibration cycles at the stress level $\sigma(vib)_i$.

$$D = \sum_{i=1}^{2} D_2 = \left[\frac{n_2}{N_{2,f}}\right]^{\left(\frac{N_{2,f}}{N_{1,f}}\right)^{\left[\frac{\sigma(vib)_1}{\sigma(vib)_2}\right]}} \tag{5}$$

Now that the random fatigue damage generated by the vibration environment is determined, a Weibull formulation is presented that let us use the generated damage in the Weibull Y vector (see Equation (13)) to determine the reliability of the analyzed element.

2.2. Weibull Analysis

A two-parameter Weibull distribution is used to statistically analyze fatigue behaviors [23–25]. It allowed us to perform accurate fatigue failure analysis [26,27]. The probability density function $f(t)$ and cumulative distribution function $F(t)$ are described by Equations (6) and (7), respectively.

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} \tag{6}$$

$$F(t) = 1 - exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} \tag{7}$$

where, β is the shape parameter, η is the scale parameter, and t is the selected random variable (damage or fatigue life). The corresponding reliability function $R(t)$ is given as

$$R(t) = exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\} \tag{8}$$

From [28], the Weibull fatigue damage β_D and η_D parameters are determined as

$$\beta_D = \frac{-4\mu_y}{0.995 \times ln\left(\frac{\sigma_1}{\sigma_2}\right)} \tag{9}$$

$$\eta_D = exp(\mu_x) \tag{10}$$

where μ_y represents the mean of the Y vector (see Equation (13)) determined by using Equation (5). μ_x represents the log-mean of the failure-time data, which is determined here directly from the addressed maximum σ_1 and minimum σ_2 stress values of Section 2.1. Thus, μ_x is determined as

$$\mu_x = ln(\sigma_1\sigma_2)^{\frac{1}{2}} \tag{11}$$

Here, notice that the efficiency of the Weibull parameters β_D and η_D only depends on the accuracy with which the σ_1 and σ_2 values are determined by Equation (1). In this paper, they were determined from an electrodynamic shaker acceleration response and by the method performed in [17], Section 3.1.

2.3. Weibull Fatigue Damage Analysis

The expected behavior of the σ_1 and σ_2 parameters is determined by the following steps:

Step 1. By using the fatigue damage accumulation results from the component's operation, the times that the vibration profile (PSD) loading were applied to the mechanical component are taken as n.

Step 2. The accumulation fatigue damage D_i determined by Equation (5) in the end of each one of the n vibration loadings is used by Equation (12) as the corresponding cumulated failure percentile $R(D_i)$, as

$$R(D_i) = 1 - D_i \tag{12}$$

Step 3. By using the $R(D_i)$ elements in the linearized form of the reliability function given in Equation (8), the corresponding Y_i elements are determined as in Equation (13). Then, its corresponding arithmetic mean value is computed as in Equation (14).

$$Y_i = LN(-LN(1 - D_i)) \tag{13}$$

$$\mu_y = \sum_{i=1}^{n} \frac{Y_i}{n} \tag{14}$$

Step 4. By plugging the μ_y value and the σ_1 and σ_2 values into Equation (9), the corresponding Weibull shape β_D parameter is determined. Similarly, by plugging the σ_1 and σ_2 values into Equation (11), the corresponding μ_x value is determined. Then, by using this corresponding μ_x value in Equation (10), the corresponding Weibull scale η_D parameter is determined. These β_D and η_D parameters represent the Weibull fatigue damage family that is used to model the random behavior of the estimated σ_1 and σ_2 principal stress values.

Note 1. Here, notice the random behavior of the σ_1 and σ_2 values. In the proposed Weibull analysis, let us use the $\sigma_{1\mathrm{i}}$ values as the minimum required strength that the component's material must present in order to ensure that the reliability of the component will meet at least (as a minimum) the desired $R(t)$ index.

From the Weibull analysis, by using the β_D and η_D parameters, the minimum strength $\sigma_{1\mathrm{i}}$ values are determined by using the t_{0i} value that correspond to each Y_i element as

$$t_{0i} = \exp\{Y_i / \beta_D\} \tag{15}$$

Thus, the $\sigma_{2\mathrm{i}}$ value is determined as

$$\sigma_{2i} = \eta_D \times t_{0i} \tag{16}$$

and the $\sigma_{1\mathrm{i}}$ value is determined as

$$\sigma_{1i} = \eta_D / t_{0i} \tag{17}$$

Additionally, from Equation (18), by using the known σ_1 value, the t_{01} element that belongs to the σ_1 and σ_2 values determined in Section 2.1 is determined as

$$t_{01} = \eta_D / \sigma_1 \tag{18}$$

Now, the t_{01} and the β_D values are used to determine the corresponding Y_1 value as follows,

$$Y_1 = \ln(t_{01}) \times \beta_D \tag{19}$$

Finally, the reliability index that corresponds to the Y_1 value is determined as

$$R(t) = \exp\{-\exp\{Y_1\}\} \tag{20}$$

Note 2. Here, observe that the $R(t)$ index determined in Equation (20) by using the σ_1 value according to our proposed method corresponds to a component with strength equivalent to the σ_1 value. Thus, if we define the material S_y parameter as the actual strength of the component, then by using this S_y value in Equation (18) and the corresponding Y_{Sy} value of Equation (19) in Equation (20), the minimum expected reliability of a component that presents a strength of S_y is determined. Please also notice from the proposed Weibull analysis that any desired strength value can be used to determine its corresponding reliability. The diagram with the steps required to determine the probability of failure $F(t)$ and the reliability $R(t)$ based on the vibration fatigue damage D is shown in Figure 1.

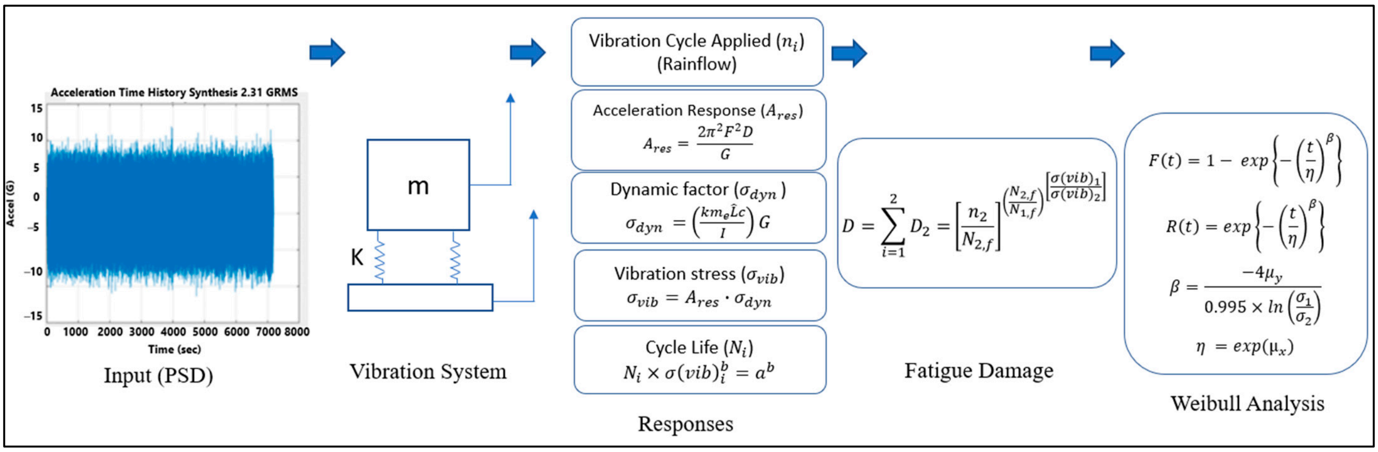

Figure 1. Diagram of the Weibull Fatigue Damage Analysis.

Next, a numerical application is presented where the proposed method is applied to a mechanical component that is submitted to random vibration stress due to its field application.

3. Numerical Application

3.1. Fatigue Damage Accumulation

The accumulation fatigue damage study case is performed by analyzing a mechanical support component that is used to install a fiber optic panel into a frame. It is shown in Figure 2.

The support is made of cold drawn steel AISI 1025, with a modulus of elasticity E = 200 GPa, Poisson's ratio r = 0.29, yield strength S_y = 430 MPa, ultimate tensile strength S_{ut} = 510 MPa, endurance limit S_e = 255 MPa, density ρ = 7.9 g/cm^3, length L = 51 mm, width W = 200 mm, and a wall thickness t = 3 mm. During its function, the component supports a static load of 80 N. It is submitted to an operating random vibration with an input PSD consisting of frequencies ranging from 10 to 55 Hz at an amplitude of 1.5 mm peak to peak for a period of 2 h. The testing is carried out physically by using a vibration system, and the results are as follows. By using Equations (1) and (2), the dynamic factor $\sigma_{dynamic}$ and the vibration stresses $\sigma(vib)_i$, are calculated, respectively. The acceleration responses $A_{res\,i}$ are obtained from the vibration system but can also be determined by using Equation (3). Then, the vibration cycles applied n_i are determined by the rainflow method and the total cycles N_i are determined by using Equation (4). Finally, the fatigue damage accumulation is obtained by Equation (5). Table 1 shows the vibration stress and cycle results and Table 2 shows the vibration fatigue damage accumulation results, where the failure is presented when $D = 1$ [29].

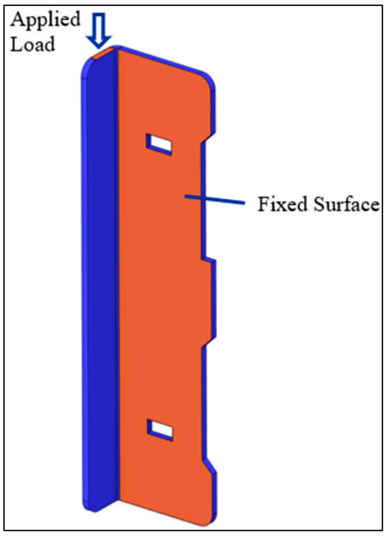

Figure 2. Mechanical support.

Table 1. Vibration Stress and Cycles Results.

Frequency (Hz)	Accel. Response (G)	Dynamic Factor $\sigma_{dynamic}$ Equation (2)	Vibration Stress $\sigma(vib)_i$ Equation (1)	Applied Vibration Cycles (n_i)	Total Cycles (N_i) Equation (4)
10	0.72	22.22	15.99	70,384	1.36×10^{20}
20	2.65		58.86	140,195	3.04×10^{13}
30	5.62		124.84	92,619	4.42×10^{9}
40	9.17		203.69	10,807	1.40×10^{7}
50	13.72		304.76	2921	1.23×10^{5}
55	12.36		274.55	762	4.20×10^{5}

Figure 3 shows the mechanical support areas where the fatigue damage accumulated was presented. The analysis and estimation of the damage were determined from the acceleration responses that were the base to calculate the principal vibration stresses, σ_1 and σ_2, that are employed in the Weibull probabilistic analysis.

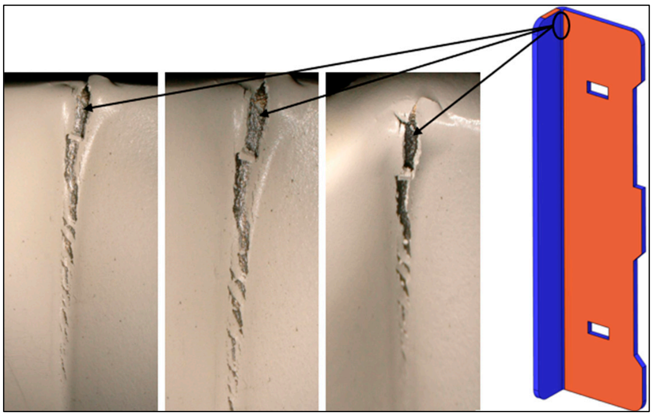

Figure 3. Fatigue cracks on the support after vibration testing.

Table 2. Vibration Fatigue Damage Accumulation.

	10 Hz	20 Hz	30 Hz	40 Hz	50 Hz	55 Hz
Block No.	D_1	D_{1+2}	D_{1+2+3}	$D_{1+2+3+4}$	$D_{1+2+3+4+5}$	$D_{1+2+3+4+5+6}$
1 (2 h)	5.18×10^{-16}	5.18×10^{-16}	5.20×10^{-16}	5.73×10^{-16}	2.37×10^{-2}	2.00×10^{-2}
2 (2 h)	2.00×10^{-2}	2.42×10^{-2}	2.42×10^{-2}	2.42×10^{-2}	4.84×10^{-2}	5.00×10^{-2}
3 (2 h)	5.00×10^{-2}	4.91×10^{-2}	4.93×10^{-2}	5.01×10^{-2}	7.38×10^{-2}	7.00×10^{-2}
4 (2 h)	7.00×10^{-2}	7.49×10^{-2}	7.51×10^{-2}	7.63×10^{-2}	1.00×10^{-1}	1.00×10^{-1}
5 (2 h)	1.00×10^{-1}	1.01×10^{-1}	1.02×10^{-1}	1.03×10^{-1}	1.27×10^{-1}	1.30×10^{-1}
6 (2 h)	1.30×10^{-1}	1.29×10^{-1}	1.29×10^{-1}	1.31×10^{-1}	1.55×10^{-1}	1.60×10^{-1}
7 (2 h)	1.60×10^{-1}	1.56×10^{-1}	1.57×10^{-1}	1.59×10^{-1}	1.83×10^{-1}	1.90×10^{-1}
8 (2 h)	1.90×10^{-1}	1.85×10^{-1}	1.86×10^{-1}	1.89×10^{-1}	2.12×10^{-1}	2.10×10^{-1}
9 (2 h)	2.10×10^{-1}	2.14×10^{-1}	2.15×10^{-1}	2.18×10^{-1}	2.42×10^{-1}	2.40×10^{-1}
10 (2 h)	2.40×10^{-1}	2.45×10^{-1}	2.45×10^{-1}	2.49×10^{-1}	2.73×10^{-1}	2.80×10^{-1}
11 (2 h)	2.80×10^{-1}	2.76×10^{-1}	2.76×10^{-1}	2.81×10^{-1}	3.04×10^{-1}	3.10×10^{-1}
12 (2 h)	3.10×10^{-1}	3.07×10^{-1}	3.08×10^{-1}	3.13×10^{-1}	3.37×10^{-1}	3.40×10^{-1}
13 (2 h)	3.40×10^{-1}	3.40×10^{-1}	3.41×10^{-1}	3.46×10^{-1}	3.70×10^{-1}	3.70×10^{-1}
14 (2 h)	3.70×10^{-1}	3.73×10^{-1}	3.74×10^{-1}	3.80×10^{-1}	4.03×10^{-1}	4.10×10^{-1}
15 (2 h)	4.10×10^{-1}	4.07×10^{-1}	4.08×10^{-1}	4.14×10^{-1}	4.38×10^{-1}	4.40×10^{-1}
16 (2 h)	4.40×10^{-1}	4.42×10^{-1}	4.43×10^{-1}	4.50×10^{-1}	4.73×10^{-1}	4.80×10^{-1}
17 (2 h)	4.80×10^{-1}	4.77×10^{-1}	4.79×10^{-1}	4.86×10^{-1}	5.10×10^{-1}	5.10×10^{-1}
18 (2 h)	5.10×10^{-1}	5.14×10^{-1}	5.15×10^{-1}	5.23×10^{-1}	5.47×10^{-1}	5.50×10^{-1}
19 (2 h)	5.50×10^{-1}	5.51×10^{-1}	5.53×10^{-1}	5.61×10^{-1}	5.85×10^{-1}	5.90×10^{-1}
20 (2 h)	5.90×10^{-1}	5.89×10^{-1}	5.91×10^{-1}	6.00×10^{-1}	6.24×10^{-1}	6.30×10^{-1}
21 (2 h)	6.30×10^{-1}	6.29×10^{-1}	6.30×10^{-1}	6.40×10^{-1}	6.63×10^{-1}	6.70×10^{-1}
22 (2 h)	6.70×10^{-1}	6.68×10^{-1}	6.70×10^{-1}	6.80×10^{-1}	7.04×10^{-1}	7.10×10^{-1}
23 (2 h)	7.10×10^{-1}	7.09×10^{-1}	7.11×10^{-1}	7.22×10^{-1}	7.45×10^{-1}	7.50×10^{-1}
24 (2 h)	7.50×10^{-1}	7.51×10^{-1}	7.53×10^{-1}	7.64×10^{-1}	7.88×10^{-1}	7.90×10^{-1}
25 (2 h)	7.90×10^{-1}	7.94×10^{-1}	7.96×10^{-1}	8.08×10^{-1}	8.31×10^{-1}	8.40×10^{-1}
26 (2 h)	8.40×10^{-1}	8.37×10^{-1}	8.40×10^{-1}	8.52×10^{-1}	8.76×10^{-1}	8.80×10^{-1}
27 (2 h)	8.80×10^{-1}	8.82×10^{-1}	8.85×10^{-1}	8.97×10^{-1}	9.21×10^{-1}	9.30×10^{-1}
28 (2 h)	9.30×10^{-1}	9.28×10^{-1}	9.30×10^{-1}	9.44×10^{-1}	9.67×10^{-1}	9.70×10^{-1}
29 (2 h)	9.70×10^{-1}	9.74×10^{-1}	9.77×10^{-1}	9.91×10^{-1}	1.01×10^{0}	1.02×10^{0}

Now, in the following section, the probabilistic approach is applied to the vibration fatigue damage results obtained in Table 2.

3.2. Weibull Fatigue Damage Analysis

Here, it is remarked that if a different sample with the same features was submitted to the same vibration damage accumulation experiment, due to random behavior, the results would be different. For the damage data given in Table 2, where 29 blocks were tested, the probabilistic analysis is as follows. For this purpose, the vector Y that includes the fatigue damage as described by Equation (13) is used, instead of using the median rank approach, to determine the Weibull parameters as follows [30,31].

By selecting the fatigue damage accumulated D_i value of each individual block of Table 2, and using it in Equation (13), the n = 29 Y_i elements are determined. Thus, from

Equation (14), the mean is, $\mu_y = -0.6672$. Next, from the results of Table 1, the principal vibration stress values σ_1 = 304.76 MPa and σ_2 = 15.99 MPa are obtained. With these data, we can proceed to use the Weibull distribution in order to obtain its failure probability, reliability, and random behavior. From Equation (9), the Weibull shape parameter β_D value is

$$\beta_D = \frac{(-4)(-0.6672)}{0.995 \times \ln\left(\frac{304.76}{15.99}\right)} = 0.9102$$

and from Equation (11), the logarithm average value is $\mu_x = \ln\sqrt{304.76 \times 15.99} = 4.2457$. Thus, from Equation (10), the Weibull parameter η_D is $\eta_D = \exp\{4.2457\} = 69.8077$. Consequently, the Weibull damage family is W(0.9102, 69.8077 MPa). This Weibull family completely represents the observed principal vibration stresses σ_1 and σ_2 values.

Now, by using the Weibull family results, the random behavior strength can be determined. Since the Weibull parameters only depend on the principal stress values caused by the random vibration σ_1 and σ_2 values, the random behavior can be obtained by performing the Weibull analysis using the following steps.

Since the determination of the fatigue damage is based on the random behavior of the σ_2 stress value, here, the random behavior of σ_1 and σ_2 is determined by Equations (16) and (17). Then, by using the β_D and the Y_i values in Equation (15), the basic Weibull elements [28] t_{0i} for each Y_i are obtained. Whereas by using the η_D and σ_1 values in Equation (18), the Weibull t_{01} value from the σ_1 and σ_2 stress values are reproduced. This value is calculated as

$$t_{01} = \frac{69.8077}{304.76} = 0.2291$$

and, by using the β_D value in Equation (19), the Y_1 value that belongs to the t_{01} value is determined as

$$Y_1 = \ln(0.2291) \times 0.9102 = -1.3414$$

Next, by substituting the Y_1 value in Equation (20), the reliability $R(t)$ that belongs to the t_{01} element is

$$R(t) = \exp\{-\exp\{-1.3414\}\} = 0.7699$$

The previous results shown in this section are included in Table 3.

Here, is important to mention that the reliability obtained $R(t) = 0.7699$ corresponds to a designed component or structure with minimal strength of S_y = σ_1 = 304.76 MPa. In relation to the mechanical support, it has a S_y = 430 MPa, then, its reliability is $R(t) = 0.8260$, which is determined as follows. The minimal reliability of the component is obtained when the Y_1 value that belongs to the S_y value is used in Equation (20). The steps to determine the reliability of the design component when the S_y value is used as σ_1 in Equation (18) are, the t_{Sy} element that belongs to the S_y value is $t_{sy} = \frac{69.8077}{430} = 0.1623$. From Equation (19), the corresponding Y_{sy} value is $Y_{sy} = \ln(0.1623) \times 0.9102 = -1.6548$. The reliability index for the Y_{sy} value is calculated by using Equation (20), $R(t) = \exp\{-\exp\{-1.6548\}\} = 0.8260$. Thus, we conclude that the reliability of the design component is $R(t) = 0.8260$. Additionally, regarding the material's component or structure, it is noticed about that the higher the strength S_y value, the higher the reliability $R(t)$ will be [28].

Now, as a comparison for the probabilistic cumulative density, the Weibull distribution is used, but in this case, by using the median rank method to estimate the vector Y, which is the base to determine the reliability and the probability of failure.

Table 3. Weibull vibration fatigue damage statistics analysis for the numerical application data.

n_i	Damage (D_i) Equation (5)	R (D_i) Equation (12)	Y_i Equation (13)	μ_y Equation (14)	$R(t)$ Equation (20)	t_{oi} Equation (15)	σ_{2i} Equation (16)	σ_{1i} Equation (17)	$F(t)$ Equation (7)
1	0.0242	0.9758	−3.7106	−0.1280	0.9758	0.0170	1.1843	4114.9115	0.0242
2	0.0491	0.9509	−2.9881	−0.1030	0.9509	0.0375	2.6192	1860.5420	0.0491
	0.0500	**0.9500**	**−2.9702**	**−0.1024**	**0.9500**	**0.0383**	**2.6712**	**1824.3190**	**0.0500**
3	0.0749	0.9251	−2.5535	−0.0881	0.9251	0.0605	4.2219	1154.2363	0.0749
4	0.1013	0.8987	−2.2366	−0.0771	0.8987	0.0857	5.9806	814.8266	0.1013
5	0.1285	0.8715	−1.9838	−0.0684	0.8715	0.1131	7.8951	617.2292	0.1285
	0.1740	**0.8260**	**−1.6548**	**−1.8180**	**0.8260**	**0.1623**	**11.3328**	**430.0000**	**0.1740**
6	0.1564	0.8436	−1.7713	−0.0611	0.8436	0.1428	9.9713	488.7123	0.1564
7	0.1851	0.8149	−1.5863	−0.0547	0.8149	0.1750	12.2184	398.8335	0.1851
8	0.2145	0.7855	−1.4212	−0.0490	0.7855	0.2098	14.6488	332.6637	0.2145
	0.2301	**0.7699**	**−1.3414**	**−1.4738**	**0.7699**	**0.2291**	**15.9900**	**304.7600**	**0.2301**
9	0.2446	0.7554	−1.2709	−0.0438	0.7554	0.2475	17.2778	282.0444	0.2446
10	0.2756	0.7244	−1.1321	−0.0390	0.7244	0.2883	20.1243	242.1505	0.2756
11	0.3072	0.6928	−1.0023	−0.0346	0.6928	0.3325	23.2107	209.9507	0.3072
12	0.3397	0.6603	−0.8794	−0.0303	0.6603	0.3805	26.5641	183.4471	0.3397
13	0.3729	0.6271	−0.7621	−0.0263	0.6271	0.4329	30.2170	161.2704	0.3729
14	0.4069	0.5931	−0.6492	−0.0224	0.5931	0.4900	34.2089	142.4515	0.4069
15	0.4418	0.5582	−0.5396	−0.0186	0.5582	0.5528	38.5881	126.2854	0.4418
16	0.4774	0.5226	−0.4323	−0.0149	0.5226	0.6219	43.4144	112.2465	0.4774
17	0.5139	0.4861	−0.3266	−0.0113	0.4861	0.6985	48.7630	99.9346	0.5139
18	0.5513	0.4487	−0.2215	−0.0076	0.4487	0.7840	54.7301	89.0389	0.5513
19	0.5895	0.4105	−0.1162	−0.0040	0.4105	0.8802	61.4414	79.3132	0.5895
20	0.6285	0.3715	−0.0097	−0.0003	0.3715	0.9894	69.0649	70.5585	0.6285
	0.6321	**0.3679**	**0.0000**	**0.0000**	**0.3679**	**1.0000**	**69.8065**	**69.8065**	**0.6321**
21	0.6685	0.3315	0.0990	0.0034	0.3315	1.1150	77.8327	62.6101	0.6685
22	0.7094	0.2906	0.2116	0.0073	0.2906	1.2617	88.0781	55.3272	0.7094
23	0.7511	0.2489	0.3299	0.0114	0.2489	1.4368	100.3032	48.5838	0.7511
24	0.7938	0.2062	0.4569	0.0158	0.2062	1.6519	115.3161	42.2588	0.7938
25	0.8375	0.1625	0.5972	0.0206	0.1625	1.9273	134.5426	36.2199	0.8375
26	0.8821	0.1179	0.7599	0.0262	0.1179	2.3045	160.8711	30.2920	0.8821
27	0.9277	0.0723	0.9659	0.0333	0.0723	2.8898	201.7269	24.1570	0.9277
28	0.9743	0.0257	1.2979	0.0448	0.0257	4.1616	290.5145	16.7741	0.9743
	0.9782	**0.0218**	**1.3414**	**1.4738**	**0.0218**	**4.3657**	**304.7600**	**15.9900**	**0.9782**
29	0.9900	0.0100	1.5272	0.0527	0.0100	5.3540	373.7498	13.0384	0.9900
	β = 0.9102	η = 69.8065	μ_y = −0.6672	σ_1 = 304.7600	σ_2 = 15.9900				

BOLD: The principal vibration stresses above and below the η parameter, the $R(t)$ index of 95%, and the $R(t)$ index of the S_y 430 MPa.

4. Median Rank Approach

In this section, the median rank method [30] is applied to the data given in Section 3. By using this method, the corresponding cumulated failure percentile $F(t_i)$ that previously was determined by Equation (12) is now determined as

$$F(t_i) = \frac{i - 0.3}{n + 0.4} \tag{21}$$

Then, by using the $F(t_i)$ elements in the linearized form of the reliability function given in Equation (8), the corresponding Y_i elements are determined as in Equation (22),

$$Y_i = LN(-LN(1 - ((i - 0.3)/(n + 0.4)))) \tag{22}$$

By selecting the fatigue damage accumulated D_i value of each one block of Table 2 and using it in Equation (22), the n = 29 Y_i elements are determined. Thus, from Equation (14), the mean is, $\mu_y = -0.5525$. Next, from the results of Table 1, the principal vibration stress values σ_1 = 304.76 MPa and σ_2 = 15.99 MPa are obtained. With these data, we can now proceed to use the Weibull distribution in order to obtain its failure probability, reliability, and random behavior. From Equation (9), the Weibull shape parameter β_D value is

$$\beta_D = \frac{(-4)(-0.5525)}{0.995 \times \ln\left(\frac{304.76}{15.99}\right)} = 0.7538$$

Since the principal vibration stress values σ_1 = 304.76 MPa and σ_2 = 15.99 MPa are maintained, the Weibull parameter value $\eta_D = 69.8077$ remains.

Next, the remaining steps followed in Section 3 are applied and the results are included in Table 4.

Table 4. Weibull vibration fatigue damage statistics analysis by using median rank method.

n_i	Y_i Equation (21)	μ_y Equation (14)	$R(t)$ Equation (20)	t_{oi} Equation (15)	σ_{2i} Equation (16)	σ_{1i} Equation (17)	$F(t)$ Equation (7)
1	−3.7256	−0.1285	0.9762	0.0071	0.4981	9783.4030	0.0238
	−2.9702	**−0.1024**	**0.9500**	**0.0194**	**1.3570**	**3591.0886**	**0.0500**
2	−2.8207	−0.0973	0.9422	0.0237	1.6546	2945.1760	0.0578
3	−2.3400	−0.0807	0.9082	0.0449	3.1311	1556.3556	0.0918
4	−2.0062	−0.0692	0.8741	0.0698	4.8755	999.5045	0.1259
5	−1.7476	−0.0603	0.8401	0.0984	6.8706	709.2668	0.1599
6	−1.5347	−0.0529	0.8061	0.1305	9.1130	534.7438	0.1939
	−1.3704	**−1.8180**	**0.7757**	**0.1623**	**11.3328**	**430.0000**	**0.2243**
7	−1.3524	−0.0466	0.7721	0.1663	11.6070	419.8408	0.2279
8	−1.1918	−0.0411	0.7381	0.2058	14.3630	339.2823	0.2619
	−1.1109	**−1.4738**	**0.7194**	**0.2291**	**15.9900**	**304.7500**	**0.2806**
9	−1.0474	−0.0361	0.7041	0.2492	17.3959	280.1293	0.2959
10	−0.9154	−0.0316	0.6701	0.2969	20.7257	235.1238	0.3299
11	−0.7930	−0.0273	0.6361	0.3492	24.3773	199.9034	0.3639
12	−0.6784	−0.0234	0.6020	0.4066	28.3815	171.7005	0.3980
13	−0.5699	−0.0197	0.5680	0.4695	32.7756	148.6811	0.4320
14	−0.4663	−0.0161	0.5340	0.5387	37.6055	129.5852	0.4660
15	−0.3665	−0.0126	0.5000	0.6149	42.9271	113.5207	0.5000
16	−0.2697	−0.0093	0.4660	0.6992	48.8095	99.8395	0.5340
17	−0.1751	−0.0060	0.4320	0.7927	55.3389	88.0595	0.5680
18	−0.0819	−0.0028	0.3980	0.8971	62.6241	77.8153	0.6020
	0.0000	**0.0000**	**0.3679**	**1.0000**	**69.8065**	**69.8065**	**0.6321**
19	0.0107	0.0004	0.3639	1.0143	70.8051	68.8243	0.6361
20	0.1033	0.0036	0.3299	1.1469	80.0653	60.8642	0.6701
21	0.1969	0.0068	0.2959	1.2986	90.6512	53.7567	0.7041
22	0.2925	0.0101	0.2619	1.4741	102.9040	47.3559	0.7381
23	0.3913	0.0135	0.2279	1.6805	117.3143	41.5390	0.7721
24	0.4950	0.0171	0.1939	1.9285	134.6219	36.1985	0.8061

Table 4. *Cont.*

n_i	Y_i Equation (21)	μ_y Equation (14)	$R(t)$ Equation (20)	t_{oi} Equation (15)	σ_{2i} Equation (16)	σ_{1i} Equation (17)	$F(t)$ Equation (7)
25	0.6062	0.0209	0.1599	2.2349	156.0158	31.2347	0.8401
26	0.7288	0.0251	0.1259	2.6298	183.5827	26.5445	0.8741
27	0.8703	0.0300	0.0918	3.1730	221.4969	22.0008	0.9082
28	1.0474	0.0361	0.0578	4.0133	280.1599	17.3940	0.9422
	1.1109	**1.4738**	**0.0480**	**4.3656**	**304.7500**	**15.9900**	**0.9520**
29	1.3185	0.0455	0.0238	5.7498	401.3795	12.1409	0.9762

BOLD: The principal vibration stresses above and below the η parameter, the $R(t)$ index of 95%, and the $R(t)$ index of the S_y 430 MPa.

A comparison of the principal results between both the proposed method and the median rank method by following the formulation given in Section 2 is shown in Table 5.

Table 5. The proposed method and median rank method comparison results.

Feature	Proposed Method	Median Rank Method
Weibull Shape Parameter Equation (9)	$\beta = 0.9102$	$\beta = 0.7538$
Weibull Scale Parameter Equation (10)	$\eta = 69.8077$	$\eta = 69.8077$
Principal Stresses, Equations (17) and (16) $\sigma_1 = 304.7600$ MPa $\sigma_2 = 15.9900$ MPa Material strength, $S_y = 430$ MPa	$R(t) = 0.8260$	$R(t) = 0.7757$
$R(t) = 0.95$ Equation (8)	$\sigma_1 = 1824.3190$ MPa $\sigma_2 = 2.6712$ MPa	$\sigma_1 = 3591.0886$ MPa $\sigma_2 = 1.3570$ MPa

In Table 5, notice that although we are using the same $\sigma_1 = 304.76$, $\sigma_2 = 15.99$ and S_y values in both methods, the component's reliability $R(t)$ is different. This difference occurs because in both methods, the Weibull β parameter shape have different values. This is because the arithmetic mean of both methods is different. Thus, because in the proposed method the addressed damage completely represents the analyzed component, we conclude that the real reliability is the one given by the proposed method. Notic that because the damage is random, the proposed method is dynamic, and its efficiency depends only on the accuracy with which σ_1 and σ_2 are determined. Finally, observe from the last row of Table 5 that if $R(t) = 0.95$ is required, then from the proposed method, the maximum allowed stress is $\sigma_1 = 1824.3190$ MPa. At the same time, using the median rank method, it is $\sigma_1 = 3591.0886$ MPa, implying that by using the median rank approach, we can predict that the component will become overstressed, lowering its life.

5. Discussion

The paper presents an alternative to fit a probabilistic vibration fatigue analysis based on the use of the damage accumulated for two parameters provided by the Weibull distribution model. The main contribution of the proposed method is the probabilistic approach that the Weibull distribution function has in the mechanical industrial field. A considerable number of industrial standards and guidelines employ the Weibull model for their fatigue analysis; thus, reliability engineers are especially familiar with this model.

This paper presents a probabilistic aspect that involves fatigue experiments and the proper use of vibration fatigue damage during the investigation and design of mechanical components. The use of vibration fatigue damage as the cumulative failure percentile in the process allows the transfer to obtain an estimation of the reliability and a prediction of failure, in spite of the significant variability involved in the vibration fatigue.

Regarding the results obtained for the probabilistic prediction of reliability and cumulated failure, based on the principal vibration stress values σ_1 = 304.76 and σ_2 = 15.99, the proposed method predicts a reliability of $R(t)$ = 0.826 and a cumulative failure of $F(t)$ = 0.174. Here, it is important to notice that if we use the standard median rank approach, then $R(t)$ = 0.776 and $F(t)$ = 0.224. By their comparison, we have an $R(t)$ variation of 5%. Moreover, it can be observed that because the used cumulated damage depends, among other factors, on the vibration load distribution applied, the material's strength, and the geometry of the tested mechanical component, then because each analysis is different, the proposed method is completely dynamic, and can be used in any application where the accumulated damage can be measured. Here, we highlight that the only restriction to use the cumulated damage is that the failure must be defined when the damage is equal to one. Therefore, among other applications, the proposed model can be used in the automotive and telecommunications industries, where mechanical supports are used to hold electrical and electronic devices that are subjected to environmental vibration due to their field application.

6. Conclusions

1. A probabilistic alternative of the Weibull distribution to vibration fatigue analysis is developed, which allows it to define reliability and probability of failure.
2. Contrary to other models that use the median rank method as the cumulated failure percentile, the proposed methodology considers the accumulated fatigue damage for a platform in which the component's probabilistic life, the probability of failure, and the reliability index are estimated.
3. A methodology is developed based on the model presented to permit a probabilistic approach to vibration fatigue damage accumulation which allows the probabilistic failure and reliability estimation for mechanical components and structures subjected to variable amplitude loading, specifically random vibration.
4. The model and methodology included in this work are applied to mechanical components used in the telecommunication industry to assist in real and practical structural fatigue analysis.
5. An application case is selected to illustrate the proposed Weibull model and its parameter estimation methodology based on the cumulated vibration damage included in this paper.

Author Contributions: Conceptualization, J.M.B.-C., M.R.P.-M. and R.C.T.-V.; methodology, J.M.B.-C. and M.R.P.-M.; data analysis, J.M.B.-C. and R.C.T.-V.; writing—original draft preparation, J.M.B.-C. and M.R.P.-M.; writing—review and editing, J.M.B.-C., M.R.P.-M. and R.C.T.-V.; supervision, M.R.P.-M.; funding acquisition, J.M.B.-C. and R.C.T.-V. All authors have read and agreed to the published version of the manuscript.

Funding: The research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sosnovskiy, L. *Tribo Fatigue*; Springer: New York, NY, USA, 2005.
2. Piña-Monarrez, M.R. Weibull analysis for normal/accelerated and fatigue random vibration test. *Qual. Reliab. Eng. Int.* **2019**, *35*, 2408–2428. [CrossRef]
3. Mršnik, M.; Slavič, J.; Boltežar, M. Frequency-domain methods for a vibration-fatigue-life estimation—Application to real data. *Int. J. Fatigue* **2013**, *47*, 8–17. [CrossRef]
4. Zaitseva, E.; Levashenko, V.; Rabcan, J. A new method for analysis of Multi-State systems based on Multi-valued decision diagram under epistemic uncertainty. *Reliab. Eng. Syst. Saf.* **2023**, *229*, 108868. [CrossRef]
5. Isametova, M.E.; Nussipali, R.; Martyushev, N.V.; Malozyomov, B.V.; Efremenkov, E.A.; Isametov, A. Mathematical Modeling of the Reliability of Polymer Composite Materials. *Mathematics* **2022**, *10*, 3978. [CrossRef]
6. Martyushev, N.V.; Malozyomov, B.V.; Sorokova, S.N.; Efremenkov, E.A.; Valuev, D.V.; Qi, M. Review Models and Methods for Determining and Predicting the Reliability of Technical Systems and Transport. *Mathematics* **2023**, *11*, 3317. [CrossRef]
7. Benaroya, H.; Han, S.M. *Probability Models in Engineering and Science*; Taylor & Francis Group, LLC: Boca Raton, FL, USA, 2005.
8. Shimokawa, T.; Tanaka, S. A statistical consideration of Miner's rule. *Int. J. Fatigue* **1980**, *2*, 165–170. [CrossRef]
9. Sun, Q.; Dui, H.N.; Fan, X.L. A statistically consistent fatigue damage model based on Miner's rule. *Int. J. Fatigue* **2014**, *69*, 16–21. [CrossRef]
10. Castillo, E.; Fernandez-Canteli, A. *A Unified Statistical Methodology for Modeling Fatigue Damage*; Springer: Berlin/Heidelberg, Germany, 2009.
11. Harlow, D.G. Statistical Modeling for Fatigue Life. *Metals* **2023**, *13*, 1419. [CrossRef]
12. Pinto, J.M.A.; Pujol, J.C.F.; Cimini, C.A. Probabilistic cumulative damage model to estimate fatigue life. *Fatigue Fract. Eng. Mater. Struct.* **2014**, *37*, 85–94. [CrossRef]
13. Rinne, H. *The Weibull Distribution: A Handbook*; Taylor & Francis Group, LLC: Boca Raton, FL, USA, 2009.
14. Batsoulas, N.D. Cumulative Fatigue Damage: CDM-Based Engineering Rule and Life Prediction Aspect. *Steel Res. Int.* **2016**, *87*, 1670–1677. [CrossRef]
15. Abdullah, S.; Beden, S.M.; Ariffin, A.K.; Rahman, M.M. Fatigue Life Assessment for Metallic Structure: A Case Study of Shell Structure under Variable Amplitude Loading. *J. Appl. Sci.* **2008**, *8*, 1622–1631. [CrossRef]
16. Burhan, I.; Kim, H. S-N Curve Models for Composite Materials Characterization: An Evaluative Review. *J. Compos. Sci.* **2018**, *2*, 38. [CrossRef]
17. Barraza-Contreras, J.M.; Piña-Monarrez, M.R.; Molina, A.; Torres-Villaseñor, R.C. Random Vibration Fatigue Analysis Using a Nonlinear Cumulative Damage Model. *Appl. Sci.* **2022**, *12*, 4310. [CrossRef]
18. *ASTM E1049-85*; ASTM International Standard Practices for Cycle Counting in Fatigue Analysis. ASTM International Standard: West Conshohocken, PA, USA, 2017.
19. Usabiaga, H.; Muniz-Calvente, M.; Ramalle, M.; Urresti, I.; Fernández Canteli, A. Improving with probabilistic and scale features the Basquin linear and bi-linear fatigue models. *Eng. Fail. Anal.* **2020**, *116*, 104728. [CrossRef]
20. Irvine, T. A Fatigue Damage Spectrum Method for Comparing Power Spectral Density Base Input Specifications. Vibrationdata. 2014. Available online: https://vibrationdata.wordpress.com/ (accessed on 28 July 2023).
21. Harris, C.M.; Piersol, A.G. *Shock and Vibration Handbook*, 5th ed.; McGraw Hill: New York, NY, USA, 2002; Volume 15, No. 9. [CrossRef]
22. Manson, S.S.; Halford, G.R. Practical implementation of the double linear damage rule and damage curve approach for treating cumulative fatigue damage. *Int. J. Fract.* **1981**, *17*, 169–192. [CrossRef]
23. Weibull, W. *A Statistical Theory of the Strength of Materials*; Generalstabens Litografiska Anstalts Förlag: Stockholm, Sweden, 1939.
24. Harikrishnan, R.; Mohite, P.M.; Upadhyay, C.S. Generalized Weibull model-based statistical tensile strength of carbon fibres. *Arch. Appl. Mech.* **2018**, *88*, 1617–1636. [CrossRef]
25. Melikayeva, I.; Riesch-Oppermann, H.; Govorukha, V.B.; Kraft, O. A generalized Weibull approach to interface failure in bi-material ceramic joints. *Arch. Appl. Mech.* **2011**, *81*, 1585–1596. [CrossRef]
26. Choi, S.-K.; Grandhi, R.V.; Canfield, R.A. *Reliability-Based Structural Design*; Springer-Verlag: London, UK, 2006.
27. Barraza-Contreras, J.M.; Piña-Monarrez, M.R.; Molina, A. Fatigue-life prediction of mechanical element by using the Weibull distribution. *Appl. Sci.* **2020**, *10*, 6384. [CrossRef]
28. Piña-Monarrez, M.R. Weibull stress distribution for static mechanical stress and its stress/strength analysis. *Qual. Reliab. Eng. Int.* **2018**, *34*, 229–244. [CrossRef]
29. Lee, Y.-L.; Pan, J.; Hathaway, R.B.; Barkey, M.E. *Fatigue Testing and Analysis: Theory and Practice*; Elsevier Butter-worth-Heinemann: Burlington, MA, USA, 2005.
30. Mischke, C.R. Predicting Fatigue Endurance Strengths for High- and Low-Cycle Fatigue. *J. Mech. Des.* **1982**, *104*, 653–660. [CrossRef]
31. Dias, J.P.; Ekwaro-Osire, S.; Cunha, A., Jr.; Dabetwar, S.; Nispel, A.; Alemayehu, F.M.; Endeshaw, H.B. Parametric probabilistic approach for cumulative fatigue damage using double linear damage rule considering limited data. *Int. J. Fatigue* **2019**, *127*, 246–258. [CrossRef]