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Pre-twisted calculus and differential equations

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ABSTRACT

In this paper we introduce the *$\varphi\mathbb{A}$-differentiability* for functions $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$, where U is an open set, $\mathbb{A}$ is the linear space $\mathbb{R}^n$ endowed with a unital associative commutative algebra product, and $\varphi : U \subset \mathbb{R}^k \to \mathbb{A}$ is a differentiable function in the usual sense. We call it *pre-twisted differentiability*. With respect to the $\varphi\mathbb{A}$-differentiability we introduce: (a) a type Cauchy–Riemann equations, which serve as $\varphi\mathbb{A}$-differentiability criteria, (b) a Cauchy-integral theorem, and (c) *$\varphi\mathbb{A}$-differential equations*, which can be used to solve linear and nonlinear ODE systems. It has recently been shown that the $\varphi\mathbb{A}$-differentiable functions define a complete solutions for the PDEs of the form $Au_{xx} + Bu_{xy} + Cu_{yy} = 0$, which is used in this paper for solving the corresponding Cauchy problems. Furthermore, solutions of $\varphi\mathbb{A}$-differential equations define solutions for linear and nonlinear PDE systems.

Introduction

The differentiability in the sense of Lorch corresponds to the Fréchet differentiability with respect to algebras. There is also a Gâteaux differentiability with respect to algebras. With respect to this two definitions of differentiability there have been several works to solve classical partial differential equations (PDEs for plural and PDE for singular) of mathematical physics by means of conjugate functions of differentiable functions, see [1–8], and [9]. Ketchum's work stands out, which perhaps has not been understood or has been misinterpreted, because he works with algebras in a general way, which makes it seem a bit complicated to construct solutions of PDEs by the method he proposes. The theory of analytic functions over algebras has been developed since the end of the 19th century, also see [10–14], and [15].

In this paper the "Pre-twisted Differentiability" is introduced, which is similar to the differentiability in the sense of Lorch. We say f is $\mathbb{A}$-*differentiable* if f is differentiable in the sense of Lorch with respect to $\mathbb{A}$. Pre-twisted differentiability depends on a differentiable function in the usual sense φ and on an algebra $\mathbb{A}$ (see Section 1). Thus, we call it $\varphi\mathbb{A}$-*differentiability* to make explicit the dependence. This differentiability can be used to make more explicit the method given by Ketchum for the construction of solutions of classical mathematical physics PDEs; in [16] algebras have been given for which the conditions (2), (3), and (4), given in [4] pp. 642 are satisfied, so that their fulfillment is no longer required. Thus, only the conditions given in (47) of [4] pp. 660 are required to be satisfied, and if we consider $w = w(x)$ affine, only the second condition of (47) will be required.

A vector field f is said to be *algebrizable* if f is $\mathbb{A}$-differentiable for an algebra $\mathbb{A}$. The algebrizability of ordinary differential equations has also been defined, see [17,18], and [19]. The case of algebrizability of systems of two first order PDEs with two dependent and two independent variables is being worked in [20]. In this paper we introduce: the $\varphi\mathbb{A}$-differentiability, the generalized Cauchy–Riemann equations for the $\varphi\mathbb{A}$-differentiability ($\varphi\mathbb{A}$-CREs), a version of the Cauchy-integral theorem for the $\varphi\mathbb{A}$-differentiability, a generalization of the first fundamental Theorem of calculus, Taylor expansions, and the $\varphi\mathbb{A}$-differential equations. These allow us to consider the algebrizability of a wider class of ODEs, PDEs, and PDE systems.

We call *generalized Cauchy–Riemann equations ($\mathbb{A}$-CREs)* to the generalized Cauchy–Riemann equations for the $\mathbb{A}$-differentiability. These equations serve as a criteria for the $\mathbb{A}$-differentiability. Therefore, the $\mathbb{A}$-differentiable functions give a complete solution of the $\mathbb{A}$-CREs. The use of pre-twisted differentiability very impressively expands the class of PDE systems that can be solved in this way. The $\varphi\mathbb{A}$-CREs is a linear system of $n(k-1)$ PDEs, see Section 1.5. The $\varphi\mathbb{A}$-differentiable functions are solutions of the $\varphi\mathbb{A}$-CREs, see Theorem 1.1, and Theorem 1.2 gives conditions under which the solutions of the $\varphi\mathbb{A}$-CREs are $\varphi\mathbb{A}$-differentiable functions. Therefore, the set of all the $\varphi\mathbb{A}$-differentiable functions is the set of solutions of the $\varphi\mathbb{A}$-CREs. If $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$,

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$f = (u, v, w)$, the $\varphi\mathbb{A}$-CREs have the form

$$\begin{array}{rcl} a_{111}u_x + a_{121}v_x + a_{131}w_x + a_{112}u_y + a_{122}v_y + a_{132}w_y &=& 0 \\ a_{211}u_x + a_{221}v_x + a_{231}w_x + a_{212}u_y + a_{222}v_y + a_{232}w_y &=& 0 \\ a_{311}u_x + a_{321}v_x + a_{331}w_x + a_{312}u_y + a_{322}v_y + a_{332}w_y &=& 0 \end{array}, \tag{1}$$

where a_{ijk} are functions of (x, y, z), $u_x = \frac{\partial u}{\partial x}$ and so on. See [7] and [21] for the generalized Cauchy–Riemann equations for the $\mathbb{A}$-differentiability.

The conjugate functions of complex analytic functions are harmonic functions, and in a simply connected domain each harmonic function is the first conjugate function of a complex analytic function. This well known result has been generalized in [16]; for each PDE of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = 0, \tag{2}$$

and for each affine planar vector field φ, an algebra $\mathbb{A}$ is given with respect to which the conjugate functions of all $\varphi\mathbb{A}$-differentiable functions are solutions for this PDE. By using the generalized Cauchy–Riemann equations associated with $\varphi\mathbb{A}$-differentiability it has been shown that each solution of this PDE is a conjugate function of a $\varphi\mathbb{A}$-differentiable function. Also, solutions for 3th and 4th order PDEs are constructed; among these are the bi-harmonic, bi-wave, and bi-telegraph equations. In Section 3.1 we consider the Cauchy Problems defined by PDEs of the form (2) and conditions of the type

$$u(x,0) = \sum_{k=0}^{\infty} a_k x^k, \qquad u_y(x,0) = \sum_{k=0}^{\infty} b_k x^k. \tag{3}$$

We show how a Cauchy problem can be solved by using $\varphi\mathbb{A}$-differentiable functions. For most of the cases the method given can use $\mathbb{A}$-differentiable functions for solving the Cauchy problem considered. However, in case (3) of Theorem 3.6 it is necessity the use of $\varphi\mathbb{A}$-differentiable functions.

In [4] pp. 659, 660, analytic functions $f(w)$ where $w = w(x)$ and f is differentiable in the sense of Lorch, are considered. An algebra $\mathbb{A}$ whose analytic functions $f(w)$ satisfy the Laplace's equation is called *harmonic algebra*. In [5] pp. 547 it is interpreted for an algebra to be a harmonic algebra it is required that $e_1^2 + e_2^2 + e_3^2 = 0$, but this condition is necessary for the case of $w = (x, y, z)$ (in our notation $\varphi(x, y, z) = (x, y, z)$). In papers [22–26], and [9] the same condition is required. This condition is used in [27] for the construction of solutions for the three dimensional Laplace's equation. Given a three dimensional vector field φ, we say that a three dimensional algebra $\mathbb{A}$ is a *φ-harmonic algebra* if

$$\varphi(e_1)^2 + \varphi(e_2)^2 + \varphi(e_3)^2 = 0. \tag{4}$$

For φ-harmonic algebras $\mathbb{A}$ the conjugate functions of all the $\varphi\mathbb{A}$-differentiable functions are harmonic functions.

On one hand we have that there does not exist any three-dimensional harmonic algebra with unit, see [28]. Secondly, all the harmonic algebras were found, see [29,30]. For the algebra $\mathbb{A}$ defined by $\mathbb{R}^3$ with the product $e_3^2 = e_2$, $e_3^3 = e_1$, where $e = e_1$ is the unit of $\mathbb{A}$, and φ given by

$$w = \varphi(x, y, z) = (-x - y + k_1, x - z + k_2, y + z + k_3),$$

one has that $e_1^2 + e_2^2 + e_3^2 = e_1 + e_2 + e_3 = (1, 1, 1)$, and identity (4) is satisfied. Therefore, $e_1^2 + e_2^2 + e_3^2 \neq 0$, and $\mathbb{A}$ is a φ-harmonic algebra.

The organization of this paper is the following. In Section 1 we recall the definition of an algebra which we denote by $\mathbb{A}$, we introduce the $\varphi\mathbb{A}$-differentiability and give some results related to this like the $\varphi\mathbb{A}$-CREs. In Section 2 the Cauchy-integral theorem for the $\varphi\mathbb{A}$-differentiability is given, the $\varphi\mathbb{A}$-differential equations and their solutions are introduced, and we show they can be used for solving linear and nonlinear ODE systems. In Section 3 we use the $\varphi\mathbb{A}$-differential functions for solving Cauchy problems for a family of PDEs and we give examples where $\varphi\mathbb{A}$-differential functions define solutions for linear and nonlinear PDE systems. In Section 4 we discuss the results obtained in this paper. It is explained how by using pre-twisted algebrizability more nonlinear ODE systems can be solved.

1. $\varphi\mathbb{A}$-Differentiability

1.1. Algebras

We call the $\mathbb{R}$-linear space $\mathbb{R}^n$ ***an algebra***; denoted by $\mathbb{A}$ if it is endowed with a bilinear product $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ denoted by $(u, v) \mapsto uv$, which is associative and commutative; that is $u(vw) = (uv)w$ and $uv = vu$ for all $u, v, w \in \mathbb{A}$; furthermore, there exists a unit $e \in \mathbb{A}$, which satisfies $eu = u$ for all $u \in \mathbb{A}$, see [31].

An element $u \in \mathbb{A}$ is called *regular* if there exists $u^{-1} \in \mathbb{A}$ called *the inverse* of u such that $u^{-1}u = e$. We also use the notation e/u for u^{-1}, where e is the unit of $\mathbb{A}$. If $u \in \mathbb{A}$ is not regular, then u is called *singular*. $\mathbb{A}^*$ denotes the set of all the regular elements of $\mathbb{A}$. If $u, v \in \mathbb{A}$ and v is regular, the quotient u/v means uv^{-1}.

The $\mathbb{A}$ product between the elements of the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ is given by $e_i e_j = \sum_{k=1}^n c_{ijk} e_k$ where $c_{ijk} \in \mathbb{R}$ for $i, j, k \in \{1, \dots, n\}$ are called *structure constants* of $\mathbb{A}$. The *first fundamental representation* of $\mathbb{A}$ is the injective linear homomorphism $R : \mathbb{A} \to M(n, \mathbb{R})$ defined by $R : e_i \mapsto R_i$, where R_i is the matrix with $[R_i]_{jk} = c_{ikj}$, for $i = 1, \dots, n$.

1.2. Definition of $\varphi\mathbb{A}$-differentiability

We use notation $x = (x_1, \dots, x_k)$. The usual differential of a function f will be denoted by df.

Let $\mathbb{A}$ be the linear space $\mathbb{R}^n$ endowed with an algebra product. Consider two differentiable functions in the usual sense $f, \varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ defined in an open set U. We say f is *$\varphi\mathbb{A}$-differentiable* on U if there exists a function $f'_\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$, that we call *$\varphi\mathbb{A}$-derivative*, such that for all $u \in U$

$$\lim_{\xi \to 0,\, \xi \in \mathbb{R}^k} \frac{f(x + \xi) - f(x) - f'_\varphi(x) d\varphi_x(\xi)}{\|\xi\|} = 0,$$

where $f'_\varphi(x) d\varphi_x(\xi)$ denotes the $\mathbb{A}$-product of $f'_\varphi(x)$ and $d\varphi_x(\xi)$. That is, f is $\varphi\mathbb{A}$-differentiable if $df_x(\xi) = f'_\varphi(x) d\varphi_x(\xi)$ for all $\xi \in \mathbb{R}^k$. In the same way, high order $\varphi\mathbb{A}$-derivatives f^n_φ can be defined by considering the limits

$$\lim_{\xi \to 0,\, \xi \in \mathbb{R}^k} \frac{f^{(n-1)}_\varphi(x + \xi) - f^{(n-1)}_\varphi(x) - f^{(n)}_\varphi(x) d\varphi_x(\xi)}{\|\xi\|} = 0.$$

We will also use the notations $\frac{df}{d\varphi}$ for f'_φ and $\frac{d^2 f}{d\varphi^2}$ for f''_φ, and so on.

If $k = n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the identity transformation $\varphi(x) = x$, the $\varphi\mathbb{A}$-differentiability will be called *$\mathbb{A}$-differentiability* and the *$\mathbb{A}$-derivative* of f will be denoted by f'. This last differentiability is known as Lorch differentiability, see [12]. Differentiability related to commutative and noncommutative algebras is considered in [10].

1.3. Algebrizability of planar vector fields

The algebrizability of planar vector fields can be see in [19]. A planar vector field $f = (u, v)$ is algebrizable on an open set $\Omega \subset \mathbb{R}^2$ if and only if the conjugate functions u, v of f satisfy at least one of the following PDE systems

(a) $u_x + p_2 v_x - v_y = 0$, $u_y - p_1 v_x = 0$,
(b) $u_x + p_1 u_y - v_y = 0$, $v_x - p_2 u_y = 0$, and
(c) $u_y = 0$, $v_x = 0$.

For case (a) we take $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$, for (b) $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$, and for (c) $\mathbb{A} = \mathbb{A}_{1,2}^2$. These systems are called *Cauchy–Riemann equations* associated with $\mathbb{A}$ ($\mathbb{A}$-CREs), where $p_i \in \mathbb{R}$ for $i = 1, \dots, 4$ are parameters, see [7]. For $\mathbb{A}_1^2(p_1, p_2)$ the product is

$\cdot$	e_1	e_2
e_1	e_1	e_2
e_2	e_2	$p_1 e_1 + p_2 e_2$

(5)

hence the unit is $e = e_1$. The structure constants are

$$\begin{array}{llll} c_{111} = 1, & c_{112} = 0, & c_{121} = 0, & c_{122} = 1, \\ c_{211} = 0, & c_{212} = 1, & c_{221} = p_1, & c_{222} = p_2, \end{array} \tag{6}$$

or equivalently, its first fundamental representation is

$$R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix}. \tag{7}$$

For $\mathbb{A}_2^2(p_1, p_2)$ the product is

$\cdot$	e_1	e_2
e_1	$p_1 e_1 + p_2 e_2$	e_1
e_2	e_1	e_2

(8)

hence the unit is $e = e_2$. The structure constants are

$$\begin{array}{llll} c_{111} = p_1, & c_{112} = p_2, & c_{121} = 1, & c_{122} = 0, \\ c_{211} = 1, & c_{212} = 0, & c_{221} = 0, & c_{222} = 1, \end{array} \tag{9}$$

or equivalently, its first fundamental representation is

$$R(e_1) = \begin{pmatrix} p_1 & 1 \\ p_2 & 0 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

For $\mathbb{A}_{1,2}^2$ the product is

$\cdot$	e_1	e_2
e_1	e_1	0
e_2	0	e_2

(11)

hence the unit is $e = e_1 + e_2$. The structure constants are

$$\begin{array}{llll} c_{111} = 1, & c_{112} = 0, & c_{121} = 0, & c_{122} = 0, \\ c_{211} = 0, & c_{212} = 0, & c_{221} = 0, & c_{222} = 1, \end{array} \tag{12}$$

or equivalently, its first fundamental representation is

$$R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

1.4. On $\varphi\mathbb{A}$-differentiability

The $\varphi\mathbb{A}$-derivative $f'_\varphi(x)$ is unique if $d\varphi_x(\mathbb{R}^k)$ is not contained in the singular set of $\mathbb{A}$. The function $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable and $\varphi'_\varphi(x) = e$ for all $x \in U$ where $e \in \mathbb{A}$ is the unit. Also, the $\mathbb{A}$-combinations (linear) and $\mathbb{A}$-products of $\varphi\mathbb{A}$-differentiable functions are $\varphi\mathbb{A}$-differentiable functions and they satisfy the usual rules of differentiation. In the same way if f is $\varphi\mathbb{A}$-differentiable and has image in the regular set of $\mathbb{A}$, then the function $\frac{e}{f^n}$ is $\varphi\mathbb{A}$-differentiable for $n \in \{1, 2, \dots\}$, and

$$\left(\frac{e}{f^n}\right)'_\varphi = \frac{-n f'_\varphi}{f^{n+1}}. \tag{14}$$

A $\varphi\mathbb{A}$-*polynomial function* $p : \mathbb{R}^k \to \mathbb{R}^n$ is defined by

$$p(x) = c_0 + c_1\varphi(x) + c_2(\varphi(x))^2 + \cdots + c_m(\varphi(x))^m \tag{15}$$

where $c_0, c_1, \dots, c_m \in \mathbb{A}$ are constants and the variable u represent the variable in $\mathbb{R}^k$. A $\varphi\mathbb{A}$-*rational function* is defined by a quotient of two $\varphi\mathbb{A}$-polynomial functions. Then, $\varphi\mathbb{A}$-*polynomial functions* and $\varphi\mathbb{A}$-*rational functions* are $\varphi\mathbb{A}$-differentiable and the usual rules of differentiation are satisfied for the $\varphi\mathbb{A}$-derivative.

In general, the rule of chain does not have sense since φ is $\varphi\mathbb{A}$-differentiable however the composition $\varphi\circ\varphi$ only is defined when $k = n$. Even in this case the rule of the chain cannot be verified. Suppose that φ is a linear isomorphism and that the rule of chain is satisfied. Thus the Jacobian matrix of $\varphi\circ\varphi$ satisfies

$$J(\varphi\circ\varphi) = MM = R((\varphi\circ\varphi)'_\varphi)M,$$

where M is the matrix associated with φ respect to the canonical basis of $\mathbb{R}^n$ and R is the first fundamental representation of $\mathbb{A}$. Then

$$J(\varphi\circ\varphi)M^{-1} = M \in R(\mathbb{A}). \tag{16}$$

Therefore, if M has determinant $\det(M) \neq 0$ and $M \notin R(\mathbb{A})$, the rule of chain is not valid for the $\varphi\mathbb{A}$-differentiability. By (16) $M \in R(\mathbb{A})$.

We have the following first version of the rule of chain.

Lemma 1.1. *If $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{A}$-differentiable with $\mathbb{A}$-derivative g', $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable, and $f(U) \subset \Omega$, then $g\circ f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a $\varphi\mathbb{A}$-differentiable function with $\varphi\mathbb{A}$-derivative*

$$(g\circ f)'_\varphi = (g'\circ f)f'_\varphi.$$

Proof. The function $g\circ f$ is differentiable in the usual sense and

$$d(g\circ f)_x(\xi) = dg_{f(x)}df_x(\xi) = g'(f(x))f'_\varphi(x)d\varphi_x(\xi). \quad \square$$

Lemma 1.1 has the following converse: each $\varphi\mathbb{A}$-differentiable function f can be expressed as $g\circ\varphi$ where g is an $\mathbb{A}$-algebrizable vector field, as we can see in the following lemma.

Lemma 1.2. *If $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism defined on an open set U and $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a is $\varphi\mathbb{A}$-differentiable on U, then there exists an $\mathbb{A}$-differentiable vector field g such that $f(x) = g\circ\varphi(x)$ for all $x \in U$, and $g'(\varphi(x)) = f'_\varphi(x)$.*

Proof. Define $g = f\circ\varphi^{-1}$, thus

$$dg_{\varphi(x)} = df_x d\varphi^{-1}_{\varphi(x)} = f'_\varphi(x)d\varphi_x d\varphi^{-1}_{\varphi(x)} = f'_\varphi(x).$$

This means that g is $\mathbb{A}$-differentiable at $\varphi(x)$ and its $\mathbb{A}$-derivative is $g'(\varphi(x)) = f'_\varphi(x)$. $\square$

We have the following proposition.

Proposition 1.1. *Let $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism defined on an open set U. The following three statements are equivalent*

(a) $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable on U.
(b) $g = f\circ\varphi^{-1}$ is $\mathbb{A}$-differentiable.
(c) f is differentiable in the usual sense on U and $Jf_x(J\varphi_x)^{-1} \in R(\mathbb{A})$ for all $x \in U$.

Proof. Suppose (a), by Lemma 1.2 we have (b).

Suppose (b), then $g = f\circ\varphi^{-1}$ is $\mathbb{A}$-differentiable. Since φ is a diffeomorphism $f = g\circ\varphi$ is differentiable in the usual sense, the rule of the chain gives $dg_{\varphi(x)} = df_x \circ d\varphi^{-1}_{\varphi(x)}$, and $Jg_{\varphi(x)} = Jf_x J\varphi^{-1}_{\varphi(x)} \in R(\mathbb{A})$. That is, (b) implies (c).

Suppose (c). Since f is differentiable in the usual sense $Jg_{\varphi(x)} = Jf_x J\varphi^{-1}_{\varphi(x)} \in R(\mathbb{A})$ implies $Jf_x = Jg_{\varphi(x)}J\varphi_x$. That is, $df_x = dg_{\varphi(x)}d\varphi_x$. Thus, f is $\varphi\mathbb{A}$-differentiable. $\square$

Corollary 1.1. *Let $f(x, y) = (u(x, y), v(x, y))$ be a vector field for which there exists a diffeomorphism $\phi(s, t) = (x(s, t), y(s, t))$ that is $\varphi = \phi^{-1}$ and suppose that some of the following conditions are satisfied:*

(a) There exist constants p_1 and p_2 such that

$$u_x x_s + u_y y_s + p_2(v_x x_s + v_y y_s) - (v_x x_t + v_y y_t) = 0,$$
$$u_x x_t + u_y y_t - p_1(v_x x_s + v_y y_s) = 0.$$

(b) There exist constants p_1 and p_2 such that

$$u_x x_s + u_y y_s + p_1(u_x x_t + u_y y_t) - (v_x x_t + v_y y_t) = 0,$$
$$v_x x_s + v_y y_s - p_2(u_x x_t + u_y y_t) = 0.$$

(c) $u_x x_t + u_y y_t = 0$ and $v_x x_s + v_y y_s = 0$.

In case (a) we take $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$, in (b) $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$, and in (c) $\mathbb{A} = \mathbb{A}_{1,2}^2$. Then f is $\varphi\mathbb{A}$-differentiable.

Proof. In the three cases the systems of partial differential equations are the generalized Cauchy–Riemann equations given in Section 1.3 for $g = f\circ\varphi^{-1}$, then g is $\mathbb{A}$-differentiable. Thus, by Proposition 1.1 f is $\varphi\mathbb{A}$-differentiable. $\square$

We also have the following second version of the rule of chain.

Lemma 1.3. *If $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is differentiable on an open set U, $g : V \subset \mathbb{R}^l \to \mathbb{R}^k$ is differentiable on an open set V with $g(V) \subset U$, and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable on U, then $h = f \circ g$ is $\phi\mathbb{A}$-differentiable on V for $\phi = \varphi \circ g$, and $h'_\phi(x) = f'_\varphi(g(x))$*

Proof. We have

$$dh_x = d(f\circ g)_x = df_{g(x)}dg_x = f'_\varphi(g(x))d\varphi_{g(x)}dg_x = f'_\varphi(g(x))d\phi_x.$$

Thus, $h'_\phi(x) = f'_\varphi(g(x))$. □

1.5. Cauchy-Riemann equations for the $\varphi\mathbb{A}$-differentiability

The canonical basis of $\mathbb{R}^k$ and $\mathbb{R}^n$ will be denoted by $\{e_1,\dots,e_k\}$ and $\{e_1,\dots,e_n\}$, respectively, according to the context of the uses it will be determined if e_i belongs to $\mathbb{R}^k$ or to $\mathbb{R}^n$. The directional derivatives of a function f with respect to a direction with respect to e_i are denoted by

$$f_{x_i} = f_{1x_i}e_1 + \cdots + f_{nx_i}e_n.$$

The *Cauchy–Riemann equations* for $(\varphi,\mathbb{A})$ ($\varphi\mathbb{A}$-*CREs*) is the linear system of $n(k-1)$ PDEs obtained from

$$d\varphi(e_j)f_{x_i} = d\varphi(e_i)f_{x_j} \tag{17}$$

for $i,j \in \{1,\dots,k\}$. For $i = 1,\dots,k$ suppose $\varphi = (\varphi_1,\dots,\varphi_n)$, then

$$d\varphi(e_i) = \varphi_{x_i} = \sum_{l=1}^{n}\varphi_{lx_i}e_l. \tag{18}$$

Since an algebra $\mathbb{A}$ is an $\mathbb{A}$-module or a module over $\mathbb{A}$, we have the following:

(1) If F is a solution of

In the following theorem the $\varphi\mathbb{A}$-CREs are given.

Theorem 1.1. *Let $f = (f_1,\dots,f_n)$ be an $\varphi\mathbb{A}$-differentiable function. Thus, the $\varphi\mathbb{A}$-CREs are given by*

$$\sum_{m=1}^{n}\sum_{l=1}^{n}(f_{mx_i}\varphi_{lx_j} - f_{mx_j}\varphi_{lx_i})C_{lmq} = 0 \tag{19}$$

for $1 \le i < j \le k$ and $q = 1,\dots,n$, which is a system of $n(k-1)$ partial differential equations.

Proof. The equalities (17) and (18) give

$$\sum_{q=1}^{n}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}(f_{mx_i}\varphi_{lx_j} - f_{mx_j}\varphi_{lx_i})C_{lmq}\right)e_q = 0. \quad \square$$

The directional derivatives of $\varphi\mathbb{A}$-differentiable functions are given in the following lemma.

Lemma 1.4. *If f is $\varphi\mathbb{A}$-differentiable, for each direction $x \in \mathbb{R}^k$ we have*

$$f_x = f'_\varphi(x)d\varphi_x. \tag{20}$$

Proof. The proof is obtained directly from the $\varphi\mathbb{A}$-differentiability of f. □

The $\varphi\mathbb{A}$-differentiability implies the $\varphi\mathbb{A}$-CREs, as we see in the following proposition.

Proposition 1.2. *Let $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a differentiable function in the usual sense on an open set U, and $k \in \{2,\dots,n\}$. Thus, if f is $\varphi\mathbb{A}$-differentiable, then $d\varphi(e_j)f_{x_i} = d\varphi(e_i)f_{x_j}$. That is, the conjugate functions of f satisfy the $\varphi\mathbb{A}$-CREs.*

Proof. By using (20) we have $f_{x_i} = f'_\varphi d\varphi(e_i)$ and $f_{x_j} = f'_\varphi d\varphi(e_j)$. Then $d\varphi(e_j)f_{x_i} = d\varphi(e_j)f'_\varphi d\varphi(e_i) = d\varphi(e_i)f'_\varphi d\varphi(e_j) = d\varphi(e_i)f_{x_j}$. □

We say *φ has an $\mathbb{A}$-regular direction ξ on U* if $\xi : U \to \mathbb{S}^1$ is a function $x \mapsto \xi_x$ such that $d\varphi_x(\xi_x)$ is a regular element of $\mathbb{A}$ for all $x \in U$, where $\mathbb{S}^1 \subset \mathbb{R}^k$ denotes the unit sphere centered at the origin. If φ has an $\mathbb{A}$-regular direction, Proposition 1.2 has a converse, as we can see in the following theorem.

Theorem 1.2. *Let $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a differentiable function in the usual sense on an open set U, and $k \in \{2,\dots,n\}$. Suppose that φ has regular directions on U. Thus, if the conjugate functions of f satisfy the $\varphi\mathbb{A}$-CREs, then f is $\varphi\mathbb{A}$-differentiable.*

Proof. Let ξ be a regular direction of φ on U. Since the conjugate functions of f satisfy the $\varphi\mathbb{A}$-CREs we have that $d\varphi(e_j)f_{x_i} = d\varphi(e_i)f_{x_j}$ for $1 \le i,j \le k$. Thus,

$$\begin{aligned} d\varphi(x)f_{x_i} &= f_{x_i}\sum_{j=1}^{n}x_j d\varphi(e_j) = \sum_{j=1}^{n}x_j d\varphi(e_j)f_{x_i} \\ &= \sum_{j=1}^{n}x_j d\varphi(e_i)f_{x_j} = \sum_{j=1}^{n}x_j f_{x_j}d\varphi(e_i) \\ &= df(x)d\varphi(e_i). \end{aligned}$$

Then, $f_{x_i} = \frac{f_\xi}{d\varphi(\xi)}d\varphi(e_i)$. We take $g_\varphi = \frac{f_\xi}{d\varphi(\xi)}$. By proof of Proposition 1.2 we have $df(x) = \sum_{i=1}^{k}x_i f_{x_i}$, and $g_\varphi d\varphi(x) = \sum_{i=1}^{k}x_i g_\varphi d\varphi(e_i)$. Under these conditions we have that $df(x) = g_\varphi d\varphi(x)$ for all $x \in \mathbb{R}^k$. That is, f is $\varphi\mathbb{A}$-differentiable and $f'_\varphi = g_\varphi$. □

We consider the following example.

Example 1.1. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x,y) = (y,x)$ and $\mathbb{A}$ the complex field $\mathbb{C}$. The CREs are given by

$$\varphi(e_2)(u,v)_x = \varphi(e_1)(u,v)_y.$$

Then

$$e_1(u_x,v_x) = e_2(u_y,v_y) = (-v_y,u_y),$$

from which we obtain the $\varphi\mathbb{A}$-CREs for the $\varphi\mathbb{A}$-differentiability

$$u_x = -v_y, \qquad v_x = u_y.$$

The function $f(x,y) = (y^2 - x^2, 2xy)$ satisfies $f(x,y) = (\varphi(x,y))^2$. In this case we have $u(x,y) = y^2 - x^2$ and $v(x,y) = 2xy$, and they satisfy the $\varphi\mathbb{A}$-CREs.

2. $\varphi\mathbb{A}$-Differential equations

2.1. The Cauchy-integral theorem for the $\varphi\mathbb{A}$-differentiability

If $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a $\varphi\mathbb{A}$-differentiable function defined in an open set U. The *$\varphi\mathbb{A}$-line integral* of f is defined by

$$\int_\gamma f\,d\varphi = \int_\gamma f(v)d\varphi(v') := \int_0^{t_1} f(\gamma(s))d\varphi(\gamma'(s))ds, \tag{21}$$

where γ is a differentiable function of t with values in U, $\gamma(0) = x_0$, $\gamma(t_1) = x$, $f(\gamma(s))d\varphi(\gamma'(s))$ represents the $\mathbb{A}$-product of $f(\gamma(s))$ and $d\varphi_{\gamma(s)}(\gamma'(s))$, and the right hand of (21) represents the usual line integral in $\mathbb{R}^n$.

A version of the Cauchy-integral theorem for the $\varphi\mathbb{A}$-line integral is given in the following theorem, see Corollary 10.11 pg. 49 of [10] for another version of the Cauchy-integral theorem relative to algebras.

Theorem 2.1. *Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a C^2-function defined on a simply-connected open set U and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ a $\varphi\mathbb{A}$-differentiable function. If γ is a closed differentiable path contained in U, then the $\varphi\mathbb{A}$-line integral (21) is equal to zero.*

Proof. We will show that $f\,d\varphi(\gamma') = \sum_{q=1}^{n}\langle G_q, \gamma'\rangle e_q$, where the G_q are n-dimensional conservative vector fields. Remember that $d\varphi(e_j) = \sum_{l=1}^{n}\varphi_{lx_j}e_l$.

The $\mathbb{A}$-product of f and $\varphi(\gamma')$ is given by

$$\begin{aligned} f\,d\varphi(\gamma') &= \left(\sum_{m=1}^{n} f_m e_m\right)\left(\sum_{j=1}^{k}\gamma'_j d\varphi(e_j)\right) = \sum_{m=1}^{n}\sum_{j=1}^{k}\sum_{l=1}^{n} f_m\gamma'_j\varphi_{lx_j}e_l e_m \\ &= \sum_{q=1}^{n}\left(\sum_{m=1}^{n}\sum_{j=1}^{k}\sum_{l=1}^{n} f_m\gamma'_j\varphi_{lx_j}c_{lmq}\right)e_q \\ &= \sum_{q=1}^{n}\left(\langle\sum_{j=1}^{k}\left(\sum_{m=1}^{n}\sum_{l=1}^{n} f_m\varphi_{lx_j}c_{lmq}\right)e_j, \sum_{j=1}^{k}\gamma'_j e_j\rangle\right)e_q, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of the vector field G_q and γ', and

$$G_q = \sum_{j=1}^{k}\left(\sum_{m=1}^{n}\sum_{l=1}^{n} f_m\varphi_{lx_j}c_{lmq}\right)e_j$$

for $q = 1,\dots,n$. By taking the exterior derivative of the dual 1-form of F_q, using the $\varphi\mathbb{A}$-CREs given by (19), and the commutativity of the second partial derivatives of the conjugate functions of φ, we show that this 1-form is exact. Therefore, G_q is a conservative vector field. □

If U is a simply connected open set containing x and x_0, Theorem 2.1 permit us to define

$$\int_{x_0}^{x} f\,d\varphi = \int_{\gamma} f\,d\varphi,$$

where γ is a differentiable function of t with values in U, $\gamma(0) = x_0$, and $\gamma(t_1) = x$.

Corollary 2.1. *Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a C^2-function on an open set U and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a $\varphi\mathbb{A}$-differentiable function. The vector fields*

$$G_q = \sum_{j=1}^{k}\left(\sum_{m=1}^{n}\sum_{l=1}^{n} f_m\varphi_{lx_j}c_{lmq}\right)e_j$$

for $q = 1,\dots,n$ are conservative, where $\varphi_{x_j} = \sum_{l=1}^{n}\varphi_{lx_j}e_l$.

Example 2.1. Consider the algebra $\mathbb{A}$ with product of the elements of the canonical basis of $\mathbb{R}^3$ given by

$\cdot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2+e_3	e_2+e_3
e_3	e_3	e_2+e_3	e_2+e_3

(22)

Let $\varphi(x,y) = (x,y,0)$. The function $f(x,y) = \varphi(x,y)^{-1}$ is $\varphi\mathbb{A}$-differentiable and

$$f(x,y) = \left(\frac{1}{x}, \frac{-xy-y^2}{x^3+2x^2y}, \frac{y^2}{x^3+2x^2y}\right).$$

Thus, the conservative vector fields G_i for $i = 1,2,3$ are given by

$$\begin{aligned} G_1 &= (f_1, 0) = \left(\frac{1}{x}, 0\right), \\ G_2 &= (f_2, f_1+f_2+f_3) = \left(\frac{-xy-y^2}{x^3+2x^2y}, \frac{x+y}{x^2+2xy}\right), \\ G_3 &= (f_3, f_2+f_3) = \left(\frac{y^2}{x^3+2x^2y}, -\frac{xy}{x^3+2x^2y}\right). \end{aligned}$$

If U is a simply-connected open set and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable on U, then the function

$$F(x) = \int_{x_0}^{x} f(v)d\varphi(v'),$$

for $x_0, x \in U$ is well defined. An $\varphi\mathbb{A}$-*antiderivative* of a function $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a function $F : U \subset \mathbb{R}^k \to \mathbb{R}^n$ whose $\varphi\mathbb{A}$-derivative is given by $F'_\varphi = f$.

For $\varphi\mathbb{A}$-polynomial functions the $\varphi\mathbb{A}$-antiderivative can be calculated in the usual way. The $\varphi\mathbb{A}$-line integral of $\varphi\mathbb{A}$-differentiable functions gives $\varphi\mathbb{A}$-antiderivatives, as we have in the following corollary which is a generalization of the fundamental theorem of calculus.

Corollary 2.2. *Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a C^2-function defined on a simply-connected open set U and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ a $\varphi\mathbb{A}$-differentiable function. If $x_0, x \in U$ and*

$$F(x) = \int_{x_0}^{x} f(v)d\varphi(v'),$$

then $F'_\varphi = f$.

Proof. We take the curve $\gamma(t) = x + t\xi$ joining x and $x+\xi$, thus $\gamma'(t) = \xi$. The rest of the proof is a consequence of Theorem 2.1. □

2.2. $\varphi\mathbb{A}$-Differential equations

Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field defined on an open set Ω. A $\varphi\mathbb{A}$-*differential equation* is

$$\frac{dw}{d\varphi} = F(w), \qquad w(\tau_0) = w_0, \tag{23}$$

finding a solution is understand as the problem of finding a $\varphi\mathbb{A}$-differentiable function $w : V_{\tau_0} \subset \mathbb{R}^k \to \mathbb{R}^n$ defined in a neighborhood V_{τ_0} of τ_0 such that $dw_\tau = F(w(\tau))d\varphi_\tau$ for all $\tau \in V_{\tau_0}$, and satisfying the initial condition $w(\tau_0) = w_0$. Thus, for a $\varphi\mathbb{A}$-ODE $w'_\varphi = F(w)$ the following notation has sense

$$\frac{dw}{d\varphi} = F(w). \tag{24}$$

We have the following Existence and Uniqueness Theorem for $\mathbb{A}$-algebrizable vector fields and $\varphi\mathbb{A}$-differential equations.

Theorem 2.2. *Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a C^2-function defined on an open set U and $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ a $\mathbb{A}$-differentiable vector field defined on an open set Ω with $\varphi(U) \subset \Omega$. For every initial condition $w_0 \in \Omega$ there exists a unique $\varphi\mathbb{A}$-differentiable function $w : V_{\tau_0} \subset \mathbb{R}^k \to \mathbb{R}^n$ with $w(\tau_0) = w_0$ and satisfying (23), where $V_{\tau_0} \subset U$ is a neighborhood of τ_0.*

Proof. Define

$$w_{n+1}(\tau) = \int_{\tau_0}^{\tau} F \circ w_n(v)d\varphi(v'), \qquad w_0(v) = w_0.$$

The constant function $w_0(v)$ is $\varphi\mathbb{A}$-differentiable. By Lemma 1.1 and Corollary 2.2 we have that $w_1(v)$ is $\varphi\mathbb{A}$-differentiable. Thus, we apply induction and show that $w_n(v)$ is $\varphi\mathbb{A}$-differentiable for all $n \in \mathbb{N}$. The remaining arguments are similar to the usual Existence and Uniqueness Theorem for ordinary differential equations. □

Let $\mathbb{A}$ be an algebra which as linear space is $\mathbb{R}^n$, and $\varphi : V \subset \mathbb{R}^k \to \mathbb{R}^n$ a differentiable function defined on open set V. Consider a function $F : \Omega \subset \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{A}$ defined on an open set Ω. We say F is $(\varphi\mathbb{A}, \mathbb{A})$-*differentiable* if $F(\tau, w)$ is $\varphi\mathbb{A}$-differentiable as a function of τ (with w being fixed) and F is $\mathbb{A}$-differentiable as a function of w (with τ being fixed). We also say F is $(\mathbb{A}, \mathbb{A})$-*differentiable* if $F(\tau, w)$ is $(\varphi\mathbb{A}, \mathbb{A})$-differentiable for the identity map $\varphi : \mathbb{A} \to \mathbb{A}$.

A non-autonomous $\varphi\mathbb{A}$-ordinary differential equation ($\varphi\mathbb{A}$-ODE) is written by

$$\frac{dw}{d\varphi} = F(\tau, w), \qquad w(\tau_0) = w_0, \tag{25}$$

where finding a solution is understood as the problem of finding a $\varphi\mathbb{A}$-differentiable function $w : V_{\tau_0} \subset \mathbb{R}^k \to \mathbb{R}^n$ defined in a neighborhood V_{τ_0} of τ_0 whose $\varphi\mathbb{A}$-derivative $\frac{dw}{d\varphi} = w'_\varphi$ satisfies

$$dw_\tau = F(\tau, w(\tau))d\varphi_\tau. \tag{26}$$

The corresponding existence and uniqueness of solutions can be stated for $(\varphi\mathbb{A}, \mathbb{A})$-differentiable functions $F = F(\tau, w)$.

2.3. Solutions for some $\varphi\mathbb{A}$-differential equations

Consider the ODE

$$\frac{dw}{dt} = f(t,w), \qquad f : U \subset \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n, \tag{27}$$

defined in the open set U. Suppose the existence of a differentiable function $\varphi : I \subset \mathbb{R} \to \mathbb{R}^n$ and an algebra structure $\mathbb{A}$ on $\mathbb{R}^n$ such that

- φ' has image in the regular set $\mathbb{A}^*$, and
- function $(t,x) \mapsto f(t,x)\varphi'^{-1}(t)$ is $(\varphi\mathbb{A},\mathbb{A})$-differentiable.

Thus, we say the ODE (27) is $(\varphi\mathbb{A},\mathbb{A})$-*algebrizable* and that the $\varphi\mathbb{A}$-ODE

$$\frac{dw}{d\varphi} = f(t,w)\varphi'^{-1}(t) \tag{28}$$

is the $(\varphi\mathbb{A},\mathbb{A})$-*algebrization* of (27).

Lemma 2.1. *Let $\varphi(t)$ and $f(t,x)$ be differentiable functions in the usual sense. Suppose that φ' has image in the regular set $\mathbb{A}^*$ and that function $(t,x) \mapsto f(t,x)\varphi'^{-1}(t)$ is $(\varphi\mathbb{A},\mathbb{A})$-differentiable. Thus, w is solution of (27) if and only if w is solution of (28).*

Proof. Let $w(t)$ be a solution of (27). We have $w' = dw$ and $\varphi' = d\varphi$. Thus,

$$dw(t) = [f(t,w(t))\varphi'^{-1}(t)]d\varphi'(t).$$

Therefore, $w(t)$ is a solution of (28).

Now, let $w(t)$ be a solution of (28). Then $w(t)$ is a $\varphi\mathbb{A}$-differentiable function, and

$$\frac{dw}{dt} = dw = \frac{dw}{d\varphi}d\varphi = (f(t,w)\varphi'^{-1}(t))\varphi'(t) = f(t,x).$$

Thus, $w(t)$ is a solution of (27). □

Given an algebra $\mathbb{A}$ which has linear space is $\mathbb{R}^n$, we say a function $F : \Omega \subset \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{A}$ is of $\mathbb{A}$-*separable variables* if $F(\tau,w) = K(\tau)L(w)$ for certain functions $K : \mathbb{R}^k \to \mathbb{A}$ and $L : \mathbb{R}^n \to \mathbb{A}$ which we call $\mathbb{A}$-*factors* of F.

The $\mathbb{A}$-line integral is defined by the $\varphi\mathbb{A}$-line integral when $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Some $\varphi\mathbb{A}$-differential equations can be solved, as we can see in the following results.

Proposition 2.1. *Consider an algebra $\mathbb{A}$, a differentiable function in the usual sense φ, and the $\varphi\mathbb{A}$-ODE*

$$\frac{dw}{d\varphi} = K(\tau)w, \tag{29}$$

where K is a $\varphi\mathbb{A}$-differentiable function with $\varphi\mathbb{A}$-antiderivative H. Thus, all the solutions of (29) are given by

$$w(\tau) = ce^{H(\tau)}, \tag{30}$$

where $c \in \mathbb{A}$ is a constant.

Proof. If we take the $\varphi\mathbb{A}$-derivative of $w(\tau)$ given in (30), we obtain that $w(\tau)$ is a solution for (29).

Let W be another solution of (29). Consider $\xi(\tau) = We^{-H(\tau)}$, then

$$\xi'_\varphi = W'_\varphi e^{-H(\tau)} + W(e^{-H(\tau)})(-K(\tau)) = (W'_\varphi - K(\tau)W)e^{-H(\tau)} = 0.$$

Thus, $\xi(\tau) = c$ is a constant in $\mathbb{A}$, and $W(\tau) = ce^{H(\tau)}$. □

In the following corollary we see a application of Proposition 2.1 in the solution of linear systems.

Corollary 2.3. *Let $\varphi : I \subset \mathbb{R} \to \mathbb{R}^n$ be a differentiable function in the usual sense and $\mathbb{A}$ be the linear space $\mathbb{R}^n$ endowed with an algebra product. The system of differential equations*

$$\frac{dw}{dt} = K(t)\varphi'(t)w, \tag{31}$$

where $K(t)$ is a $\varphi\mathbb{A}$-differentiable function with $\varphi\mathbb{A}$-antiderivative $H(t)$, has solution

$$w(t) = ce^{H(t)}, \tag{32}$$

where $c \in \mathbb{A}$ is a constant.

Proof. The $\varphi\mathbb{A}$-algebrization of linear system (31) is

$$\frac{dw}{d\varphi} = K(t)w. \tag{33}$$

Thus, by Proposition 2.1 $w(t)$ given in (32) is a solution for (33), and by Lemma 2.1 $w(t)$ is a solution of the linear system (31). □

Example 2.2. Consider the linear system

$$\begin{aligned} \tfrac{dx}{dt} &= (p_1e^{2t} - \tfrac{1}{2}\sin(2t))x + p_1e^t(p_2e^t - \cos(t) + \sin(t))y, \\ \tfrac{dy}{dt} &= e^t(p_2e^t - \cos(t) + \sin(t))x + (p_2e^t(p_2e^t - \cos(t) + \sin(t)) + p_1e^{2t} \\ &\quad -\tfrac{1}{2}\sin(2t))y. \end{aligned} \tag{34}$$

For $\varphi(t) = (-\cos(t), e^t)$ and $\mathbb{A} = \mathbb{A}_1^2(p_1,p_2)$, the non-autonomous linear system (34) can be written by

$$\frac{dw}{dt} = \varphi(t)\varphi'(t)w.$$

Thus, by Corollary 2.3 the solution of (34) is

$$w(t) = (k,l)e^{\frac{1}{2}\varphi^2(t)}. \tag{35}$$

In the following proposition we consider the general case of $\mathbb{A}$-separable variables.

Proposition 2.2. *Consider the $\varphi\mathbb{A}$-ODE*

$$\frac{dw}{d\varphi} = k(\tau)L(w), \tag{36}$$

where $k(\tau)L(w)$ is a $(\varphi\mathbb{A},\mathbb{A})$-differentiable function, where L has image contained in the regular set of $\mathbb{A}$. If w is implicitly defined by

$$\int^w \frac{dv}{L(v)} = \int^\tau K(s)d\varphi(s'), \tag{37}$$

where the left hand of (37) denotes the $\mathbb{A}$-line integral and the right hand of (37) denotes the $\varphi\mathbb{A}$-line integral, then w is a $\varphi\mathbb{A}$-differentiable function of τ which solves the $\varphi\mathbb{A}$-differential Eq. (36).

Proof. If w is implicitly defined by (37) as a function of τ, then by applying Lemma 1.1 to the left hand of (37) we calculate

$$\left(\int^w \frac{dv}{L(v)}\right)'_\varphi = \left(\frac{d}{dw}\int^w \frac{dv}{L(v)}\right)w'_\varphi = \frac{w'_\varphi}{L(w)}.$$

Since $K(\tau)$ is $\varphi\mathbb{A}$-differentiable, by Corollary 2.2 we have

$$\left(\int^\tau K(s)d\varphi(s')\right)'_\varphi = K(\tau).$$

Therefore, $w(\tau)$ is a solution of the $\varphi\mathbb{A}$-differential Eq. (36). □

If $L(w) \neq c_0 + c_1w$ for $c_0, c_1 \in \mathbb{A}$ in Proposition 2.2, then each $\varphi\mathbb{A}$-ODE has associated a nonlinear ODE system. In the following Corollary $L(w) = w^2$.

Corollary 2.4. *Consider an algebra $\mathbb{A}$, a differentiable function in the usual sense φ, and the $\varphi\mathbb{A}$-ODE*

$$\frac{dw}{d\varphi} = K(\tau)w^2 \tag{38}$$

where K is a $\varphi\mathbb{A}$-differentiable function with $\varphi\mathbb{A}$-antiderivative H. Then, by (37) $-w^{-1} = H(\tau) + c$, where $c \in \mathbb{A}$ is a constant. Thus, the solutions are given by

$$w(\tau) = \frac{-e}{H(\tau)+c}. \tag{39}$$

The following corollary gives nonlinear ODE systems which can be solved by solutions of $\varphi\mathbb{A}$-ODEs.

Corollary 2.5. *Let $\varphi : I \subset \mathbb{R} \to \mathbb{R}^n$ be a differentiable function in the usual sense and $\mathbb{A}$ the linear space $\mathbb{R}^n$ endowed with an algebra product. The nonlinear ODE system*

$$\frac{dw}{dt} = K(t)\varphi'(t)w^2, \tag{40}$$

where $K(t)$ is a $\varphi\mathbb{A}$-differentiable function with $\varphi\mathbb{A}$-antiderivative $H(t)$, has solution

$$w(t) = \frac{-e}{H(t)+c}, \tag{41}$$

where $c \in \mathbb{A}$ is a constant.

Proof. The $(\varphi\mathbb{A}, \mathbb{A})$-algebrization of the nonlinear system (40) is

$$\frac{dw}{d\varphi} = K(t)w^2. \tag{42}$$

Thus, by Corollary 2.4 $w(t)$ given in (41) is a solution for (42), and by Lemma 2.1 $w(t)$ is a solution for the system (40). □

In the following example we give a family of nonlinear ODE systems which are solved by solutions of $\varphi\mathbb{A}$-ODEs.

Example 2.3. Consider the family of nonlinear ODE systems

$$\begin{aligned}\frac{dx}{dt} &= (\alpha\alpha' + p_1\beta\beta')x^2 + 2p_1(\alpha'\beta + \alpha\beta' + p_2\beta\beta')xy \\ &\quad + p_1[(\alpha\alpha' + p_1\beta\beta') + p_2(\alpha'\beta + \alpha\beta' + p_2\beta\beta')]y^2, \\ \frac{dy}{dt} &= (\alpha'\beta + \alpha\beta' + p_2\beta\beta')x^2 + 2[(\alpha\alpha' + p_1\beta\beta') + p_2(\alpha'\beta + \alpha\beta' \\ &\quad + p_2\beta\beta')]xy \\ &\quad + [p_2(\alpha\alpha' + p_1\beta\beta') + (p_1 + p_2^2)(\alpha'\beta + \alpha\beta' + p_2\beta\beta')]y^2,\end{aligned} \tag{43}$$

where α, β are differentiable functions of t, and p_1, p_1 are real parameters. Let $\mathbb{A} = A_1^2(p_1, p_2)$ and $\varphi(t) = (\alpha(t), \beta(t))$. Thus, systems (43) can be written by

$$\frac{dw}{dt} = \varphi(t)\varphi'(t)w^2, \tag{44}$$

and by Corollary 2.5 their solutions are given by

$$w(t) = \frac{-e}{\frac{\varphi^2(t)}{2} + c}. \tag{45}$$

The solution of a first order ODE has the following version for the case of $\varphi\mathbb{A}$-derivative.

Proposition 2.3. *Consider the $\varphi\mathbb{A}$-ODE*

$$\frac{dw}{d\varphi} + P(\tau)w = G(\tau), \tag{46}$$

where P and G are $\varphi\mathbb{A}$-differentiable functions. The solution $w(\tau)$ is given by

$$w(\tau) = e^{-\int P(\tau)d\varphi(\tau')}\left(c + \int G(\tau)e^{\int P(\tau)d\varphi(\tau')}d\varphi(\tau')\right), \tag{47}$$

where $c \in \mathbb{A}$ is a constant.

3. On solutions of PDEs by using algebras

3.1. Solving a Cauchy problem by using $\varphi(\mathbb{A})$-differentiability

In this section the vector field φ has the form

$$\varphi(x, y) = (ax + by, cx + dy). \tag{48}$$

We consider the Cauchy problem given by the PDE (2) and conditions (3). We look for solutions which are components of $\varphi\mathbb{A}$-differentiable functions $w(x, y)$ having the form

$$w(x, y) = \sum_{k=0}^{\infty}(r_k, s_k)\varphi(x, y)^k, \tag{49}$$

where r_k and s_k depend on a_k, b_k, $\mathbb{A}$, and φ.

The following theorem, which is Theorem 4.1 of [16], gives φ and $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$ such that the conjugate functions of all the $\varphi\mathbb{A}$-differentiable functions are solutions of (2).

Theorem 3.1. *Consider φ given in (48) such that $Ac^2 + Bcd + Cd^2 \neq 0$, and p_1, p_2 given by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2}{Ac^2 + Bcd + Cd^2}, \qquad p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}. \tag{50}$$

Thus, for the algebra $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$ the conjugate functions of each $\varphi(\mathbb{A})$-differentiable function are solutions of the PDE (2).

The following theorem, which is similar to Theorem 4.1 of [16], gives φ and $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$ such that the conjugate functions of all the $\varphi\mathbb{A}$-differentiable functions are solutions of (2). In a similar way the following theorem can be proved.

Theorem 3.2. *Consider φ given in (48) such that $Aa^2 + Bab + Cb^2 \neq 0$, and p_1, p_2 given by*

$$p_1 = -\frac{2Aac + B(ad + bc) + 2Cbd}{Aa^2 + Bab + Cb^2}, \qquad p_2 = -\frac{Ac^2 + Bcd + Cd^2}{Aa^2 + Bab + Cb^2}. \tag{51}$$

Thus, for the algebra $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$ the conjugate functions of each $\varphi(\mathbb{A})$-differentiable function are solutions of the PDE (2).

Proof. The proof is similar to that of Theorem 4.1 of [16]. □

All the solutions of the PDE (2) given in Theorem 3.1 can be obtained by the conjugate functions of the $\varphi\mathbb{A}$-differentiable functions for certain φ and $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$, as we see in the following theorem which is Theorem 5.1 of [16].

Theorem 3.3. *Consider a linear vector field φ given in (48) such that $Ac^2 + Bcd + Cd^2 \neq 0$, p_1, p_2 given by (50), and $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$.*

(1) If $p_1(ad - bc) \neq 0$ and u is a solution of (2), then u is the first conjugate function of a $\varphi\mathbb{A}$-differentiable function.

(2) If $(ad - bc) \neq 0$ and v is a solution of (2), then v is the second conjugate function of a $\varphi\mathbb{A}$-differentiable function.

All the solutions of the PDE (2) given in Theorem 3.2 can be obtained by the conjugate functions of the $\varphi\mathbb{A}$-differentiable for certain φ and $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$, as we see in the following theorem which is similar to Theorem 5.1 of [16].

Theorem 3.4. *Consider a linear vector field φ given in (48) such that $Aa^2 + Bab + Cb^2 \neq 0$, p_1, p_2 given by (50), and $\mathbb{A} = \mathbb{A}_2^2(p_1, p_2)$.*

(1) If $p_2(ad - bc) \neq 0$ and v is a solution of (2), then v is the second conjugate function of a $\varphi\mathbb{A}$-differentiable function.

(2) If $(ad - bc) \neq 0$ and u is a solution of (2), then u is the first conjugate function of a $\varphi\mathbb{A}$-differentiable function.

Proof. The proof is similar to that of Theorem 5.1 of [16]. □

In the following theorem we use the $\mathbb{A}$-differentiability for solving a Cauchy problem.

Theorem 3.5. *For $AB \neq 0$ and $C = 0$ we set $\mathbb{A} = \mathbb{A}_2^2(-B/A, 0)$. Suppose that conditions (3) satisfy $a_{k+1} = -\frac{B}{A}\frac{b_k}{k+1}$ for $k \in \mathbb{N}$. Thus, each solution for the Cauchy problem defined by the PDE (2) and the conditions (3) is given by the first component of*

$$w(x, y) = \sum_{k=0}^{\infty}\left(r_k, s_k\right)(x, y)^k, \tag{52}$$

where $(r_0, s_0) = (a_0, s_0)$ (s_0 can take any value), $(r_1, s_1) = (b_0, a_1 + \frac{B}{A}b_0)$, and

$$(r_k, s_k) = \left(\left(-\frac{B}{A}\right)^k \frac{a_k}{2}, \left(-\frac{B}{A}\right)^{k-1}\frac{a_k}{2}\right), \qquad k \geq 2. \tag{53}$$

Proof. Let $\mathbb{A} = \mathbb{A}_2^2(-B/A, 0)$ and $\varphi(x, y) = (x, y)$. By Theorem 3.4 each solution u of the PDE (2) is the first component of a $\mathbb{A}$-differentiable function $w(x, y)$ which can be given by (49). By evaluating w in $(x, 0)$ we obtain

$$w(x,0) = \sum_{k=0}^{\infty} (r_k, s_k)(x,0)^k = (r_0, s_0) + \sum_{k=1}^{\infty} \left(-\frac{B}{A}\right)^{k-1} x^k (r_k, s_k)(1,0)$$
$$= (r_0, s_0) + \sum_{k=1}^{\infty} \left(-\frac{B}{A}\right)^{k-1} x^k \left(-\frac{B}{A} r_k + s_k, 0\right).$$

Then,

$$u(x,0) = r_0 + \sum_{k=1}^{\infty} \left(-\frac{B}{A}\right)^{k-1} \left(-\frac{B}{A} r_k + s_k\right) x^k = \sum_{k=0}^{\infty} a_k x^k.$$

We obtain the relations

$$r_0 = a_0, \qquad a_k = \left(-\frac{B}{A}\right)^{k-1} \left(-\frac{B}{A} r_k + s_k\right), \qquad k \in \mathbb{N}. \tag{54}$$

The partial derivative of w with respect to y is

$$w_y(x,y) = \sum_{k=1}^{\infty} k (r_k, s_k)(x,y)^{k-1}.$$

Now, we evaluate w_y in $(x, 0)$

$$w_y(x,0) = \sum_{k=0}^{\infty} (k+1)(r_{k+1}, s_{k+1})(x,0)^k$$
$$= (r_1, s_1) + \sum_{k=1}^{\infty} (k+1)\left(-\frac{B}{A}\right)^{k-1} x^k (r_{k+1}, s_{k+1})(1,0)$$
$$= (r_1, s_1) + \sum_{k=1}^{\infty} (k+1)\left(-\frac{B}{A}\right)^{k-1} x^k \left(-\frac{B}{A} r_{k+1} + s_{k+1}, 0\right).$$

Then,

$$u_y(x,0) = r_1 + \sum_{k=1}^{\infty} (k+1)\left(-\frac{B}{A}\right)^{k-1} \left(-\frac{B}{A} r_{k+1} + s_{k+1}\right) x^k = \sum_{k=0}^{\infty} b_k x^k.$$

We obtain the relations

$$r_1 = b_0, \qquad b_k = (k+1)\left(-\frac{B}{A}\right)^{k-1} \left(-\frac{B}{A} r_{k+1} + s_{k+1}\right), \qquad k \in \mathbb{N}. \tag{55}$$

From (54) and (55) we obtain $r_1 = b_0$, $s_1 = a_1 + \frac{B}{A} b_0$, and it is necessary that

$$a_{k+1} = -\frac{B}{A}\frac{b_k}{k+1}, \qquad k \in \mathbb{N}.$$

For the given values r_k, s_k in (53) for $k \geq 2$ the relations (54) and (55) are satisfied. □

In the following theorem we use $\mathbb{A}$-differentiability and $\varphi\mathbb{A}$-differentiability for solving Cauchy problems.

Theorem 3.6. *The solution of the Cauchy problem defined by the PDE (2) and the conditions (3) is given by a component of a pre-twisted differentiable function $w(x, y)$ of the form (49) which is given explicitly in following cases:*

(1) For $C \neq 0$, we set $\varphi(x, y) = (x, y)$, and an algebra $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$ with parameters p_1 and p_2 given by

$$p_1 = -\frac{A}{C}, \qquad p_2 = -\frac{B}{C}. \tag{56}$$

Thus, a solution of (2) is given by the second conjugate function of

$$w(x,y) = (r_0, a_0) + \sum_{k=0}^{\infty} \left(\frac{b_k}{k+1} + \frac{B}{C} a_{k+1}, a_k\right) \varphi(x,y)^{k+1}, \tag{57}$$

where r_0 can take any value.

(2) For $C \neq 0$, we set $\varphi(x, y) = (x + by, dy)$ such that $d \neq 0$, and $A + Bb + Cb^2 \neq 0$, an algebra $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$ with parameters p_1 and p_2 given by

$$p_1 = -\frac{A + Bb + Cb^2}{Cd^2}, \qquad p_2 = -\frac{Bd + 2Cbd}{Cd^2}. \tag{58}$$

Thus, a solution of (2) is given by the first conjugate function of

$$w(x,y) = (a_0, s_0) + \sum_{k=0}^{\infty} \left(a_{k+1}, -\frac{b a_{k+1}}{p_1 d} + \frac{b_k}{p_1 d(k+1)}\right) \varphi(x,y)^{k+1}, \tag{59}$$

where s_0 can take any value.

(3) For $B \neq 0$ and $|A| + |C| = 0$, we set $\varphi(x, y) = (x - y, x + y)$ and $\mathbb{A} = \mathbb{A}_1^2(1, 0)$. Thus, a solution of (2) is given by the first conjugate function of

$$w(x,y) = (a_0, s_0) + \sum_{k=0}^{\infty} \left(a_{k+1}, a_{k+1} + \frac{b_k}{k+1}\right) \varphi(x,y)^{k+1}, \tag{60}$$

where s_0 can take any value.

Proof. Suppose conditions given in (1). Thus, by Theorem 3.1 the conjugate functions of each $\varphi(\mathbb{A})$-differentiable function are solutions of the PDE that appears in (2). By (2) of Theorem 3.3 each solution $v(x, y)$ of the PDE in (2) is the second conjugate function of a $\varphi\mathbb{A}$-differentiable function $w(x, y)$ for $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$. We suppose w having the form (49). So,

$$w(x,0) = \sum_{k=0}^{\infty} (r_k, s_k)(x,0)^k = \left(\sum_{k=0}^{\infty} r_k x^k, \sum_{k=0}^{\infty} s_k x^k\right).$$

Since $(u(x, 0), v(x, 0)) = w(x, 0)$ we have $s_k = a_k$ for $k = 0, 1, \ldots$. The partial derivative of w with respect to y is

$$w_y(x,y) = \sum_{k=1}^{\infty} k(0,1)(r_k, s_k)\varphi(x,y)^{k-1}$$
$$= \sum_{k=1}^{\infty} k\left(-\frac{A}{C} s_k, r_k - \frac{B}{C} s_k\right) \varphi(x,y)^{k-1}.$$

Since $s_k = a_k$ and $(u_y(x, 0), v_y(x, 0)) = w_y(x, 0)$ we have

$$v_y(x,0) = \sum_{k=0}^{\infty} (k+1)\left(-\frac{A}{C} a_{k+1}, r_{k+1} - \frac{B}{C} a_{k+1}\right) x^k.$$

Therefore, by the condition given in (2) on the value of the derivative of the solution with respect to y $b_k = k\left(r_k - \frac{B}{C} a_k\right)$ for $k = 0, 1, 2, \ldots$. So,

$$r_{k+1} = \frac{b_k}{k+1} + \frac{B}{C} a_{k+1}. \tag{61}$$

Suppose conditions given in (2). Thus, by Theorem 3.1 the conjugate functions of each $\varphi(\mathbb{A})$-differentiable function are solutions of the PDE that appears in (2). By (1) of Theorem 3.3 each solution u of the PDE in (2) is the first conjugate function of a $\varphi\mathbb{A}$-differentiable function w for $\mathbb{A} = \mathbb{A}_1^2(p_1, p_2)$. We suppose w having the form (49). So,

$$w(x,0) = \sum_{k=0}^{\infty} (r_k, s_k)\varphi(x,0)^k = \left(\sum_{k=0}^{\infty} r_k x^k, \sum_{k=0}^{\infty} s_k x^k\right).$$

Since $(u(x, 0), v(x, 0)) = w(x, 0)$ we have $r_k = a_k$ for $k = 0, 1, \ldots$. The partial derivative of w with respect to y is

$$w_y(x,y) = \sum_{k=1}^{\infty} k(b,d)(r_k, s_k)\varphi(x,y)^{k-1}$$
$$= \sum_{k=1}^{\infty} [(bkr_k + p_1 dks_k, bks_k + dkr_k + p_2 dks_k)]\varphi(x,y)^{k-1}.$$

Since $(u_y(x, 0), v_y(x, 0)) = w_y(x, 0)$ we have

$$u_y(x,0) = \sum_{k=0}^{\infty} (k+1)(br_{k+1} + p_1 d s_{k+1}) x^k.$$

Therefore, by the condition given in (2) on the value of the derivative of the solution with respect to y $b_k = (k + 1)(br_{k+1} + p_1 d s_{k+1})$ for $k = 0, 1, 2, \ldots$. So,

$$s_{k+1} = -\frac{b a_{k+1}}{p_1 d} + \frac{b_k}{p_1 d(k+1)}. \tag{62}$$

Suppose conditions given in (3). So, we can use proof of case (2) to obtain $r_k = a_k$ for $k = 0, 1, \dots,$ and

$$s_{k+1} = \frac{b_k + (k+1)r_{k+1}}{k+1} = a_{k+1} + \frac{b_k}{k+1}. \quad \square \tag{63}$$

3.2. On $\varphi\mathbb{M}$-differentiability for matrix algebras $\mathbb{M}$

Let $\mathbb{M}$ be a commutative matrix algebra in $M_n(\mathbb{R})$ with base $\beta = \{R_1, R_2, \dots, R_l\}$. Consider two differentiable functions $F, \varphi : U \subset \mathbb{R}^k \to \mathbb{M}$ in the usual sense, where U is an open set. We say F is a $\varphi\mathbb{M}$-*differentiable function* if there exists a function $F'_\varphi : U \subset \mathbb{R}^k \to \mathbb{M}$ such that the usual differential df satisfies $dF_x = F'_\varphi(x)d\varphi_x$. That is, $dF_x(v) = F'_\varphi(x)d\varphi_x(v)$ for all $v \in \mathbb{R}^k$, where $F'_\varphi(x)d\varphi_x(v)$ denotes the product of the matrices $F'_\varphi(x)$ and $d\varphi_x(v)$.

For this definition we have $\varphi\mathbb{M}$-calculus and $\varphi\mathbb{M}$-differential equations

$$\frac{dw}{d\varphi} = H(\tau, w), \qquad H : U \times \mathbb{M} \to \mathbb{M}, \tag{64}$$

which are matrix differential equations. If $H(\tau, w)$ is $(\varphi\mathbb{M}, \mathbb{M})$--differentiable (see Section 2.2) and $H(\tau, w) = K(\tau)L(w)$, then we have the results given for $\varphi\mathbb{A}$-derivatives and $\varphi\mathbb{A}$-differential equations. For example,

$$\frac{dw}{d\varphi} = w \tag{65}$$

has the unique solution $w(\tau) = Me^{\varphi(\tau)}$ with $w(\tau_0) = Me^{\varphi(\tau_0)}$, where $M \in \mathbb{M}$ is a constant matrix.

3.3. On solutions of PDE systems

3.3.1. On solutions of PDE systems by using $\varphi\mathbb{A}$-differentiability

Now, we use $\varphi\mathbb{A}$-differentiability for the construction of solutions for linear and nonlinear systems of two first order PDEs with two dependent variables and two independent variables.

Consider a PDE system of the form

$$B_1 w_x + B_2 w_y = Aw. \tag{66}$$

In the following theorem we give conditions under which $\varphi\mathbb{A}$-differentiable functions are used for giving a complete solution for system (66).

Theorem 3.7. *Let $\mathbb{A}$ be an algebra with first fundamental representation $R : \mathbb{A} \to M_2(\mathbb{R})$ given by $R(e_i) = R_i$ such that $B_1, B_2, A \in R(\mathbb{A})$. Suppose that*

$$\phi(x, y) = (ax + by)R_1 + (cx + dy)R_2 \tag{67}$$

is such that

$$B_1(aR_1 + cR_2) + B_2(bR_1 + dR_2) = A. \tag{68}$$

Under these conditions function $w_p = R^{-1}(e^\phi)$ is a solution for the $\phi\mathbb{A}$-ODE $dw/d\phi = w$. Now, consider φ given by

$$\varphi(x, y) = yR^{-1}(B_1) - xR^{-1}(B_2). \tag{69}$$

Therefore, each solution w for (66) is given by

$$w(x, y) = e^{\phi(x,y)} f(x, y), \tag{70}$$

where $f(x, y)$ is a $\varphi\mathbb{A}$-differentiable function. Thus, a complete solution for the linear system (66) is found.

Proof. We have that $R(a)b = ab$ for all $a, b \in \mathbb{A}$, see Lemma 4.1 of [21]. Then, PDE system (66) can be written by

$$b_1 w_x + b_2 w_y = aw, \tag{71}$$

where $b_i = R^{-1}(B_i)$ for $i = 1, 2$, and $a = R^{-1}(A)$. The homogeneous systems of (66), (71) coincide and they are equivalent to the set of $\varphi\mathbb{A}$-CREs. Thus, the set of $\varphi\mathbb{A}$-differentiable functions define a complete solution of this system.

Let w be a solution of (66). Consider $\widetilde{w} = e^{-\phi(x,y)}w$. Then

$$\begin{aligned} B_1\widetilde{w}_x + B_2\widetilde{w}_y &= -B_1\phi_x e^{-\phi}w - B_1 e^{-\phi}w_x + B_2\phi_y e^{-\phi}w + B_2 e^{-\phi}w_y \\ &= -(B_1\phi_x + B_2\phi_y)e^{-\phi}w + (B_1 w_x + B_2 w_y)e^{-\phi} \\ &= -A\widetilde{w} + A\widetilde{w} = 0. \end{aligned}$$

Thus, $\widetilde{w}$ is a solution of the set of $\varphi\mathbb{A}$-CREs. Therefore, $\widetilde{w}$ is a $\varphi\mathbb{A}$-differentiable function f. Hence, $w = e^\phi f$ where f is a $\varphi\mathbb{A}$-differentiable function. $\square$

Consider a nonlinear PDE system of the form

$$B_1 w_x + B_2 w_y = Aw^2. \tag{72}$$

In the following theorem we use solutions of $\varphi\mathbb{A}$-ODE for the construction of solutions for (72).

Theorem 3.8. *Let $\mathbb{A}$ be an algebra with first fundamental representation $R : \mathbb{A} \to M_2(\mathbb{R})$ given by $R(e_i) = R_i$ such that $B_1, B_2, A \in R(\mathbb{A})$. Suppose that*

$$\varphi(x, y) = (ax + by)e_1 + (cx + dy)e_2 \tag{73}$$

is such that

$$B_1(aR_1 + cR_2) + B_2(bR_1 + dR_2) = -A. \tag{74}$$

Under these conditions $dw/d\varphi = -w^2$ is a $\varphi\mathbb{A}$-ODE with solutions given by

$$w(x, y) = \frac{e}{\varphi(x, y) + c} \tag{75}$$

which are solutions for the nonlinear PDE system (72).

Proof. Let w be given by (75). Thus,

$$\begin{aligned} B_1 w_x + B_2 w_y &= B_1(aR_1 + cR_2)\frac{-e}{(\varphi(x, y) + c)^2} \\ &\quad + B_2(bR_1 + dR_2)\frac{-e}{(\varphi(x, y) + c)^2} \\ &= -[B_1(aR_1 + cR_2) + B_2(bR_1 + dR_2)]w^2 \\ &= Aw^2. \end{aligned}$$

Therefore, w is a solution for (72). $\square$

3.3.2. On solutions of PDE systems by using $\varphi\mathbb{M}$-differentiability

In this section we will use columns of exponential function $E = e^\varphi$ for construction solutions of system (66), as it is known for linear systems with constant coefficients. The method given in this section is extended in [32].

For a matrix algebra $\mathbb{M}$ with base $\beta = \{R_1, R_2\}$ we consider the equation

$$B_1(x_1 R_1 + y_1 R_2) + B_2(x_2 R_1 + y_2 R_2) = A, \tag{76}$$

from which we obtain a linear system of four equations, and when this system has a solution (a, b, c, d), we define φ by

$$\varphi(x, y) = (ax + by)R_1 + (cx + dy)R_2. \tag{77}$$

If $\mathbb{M}$ is a matrix algebra and $\varphi : \mathbb{R}^2 \to \mathbb{M}$ is a differentiable function, then the exponential function $E = e^\varphi$ is $\varphi\mathbb{M}$-differentiable. In the following theorem we use this function for the construction of solutions for the PDE system (66).

Proposition 3.1. *Let $R_1, R_2 \in M_2(\mathbb{R})$ matrices with $R_1 R_2 = R_2 R_1$. If (a, b, c, d) is a solution of linear system (76) and φ is defined by (77), then the columns of the exponential function $E = e^\varphi$ define solutions for system (66).*

Proof. Under the conditions given we have that $B_1(aR_1 + cR_2) + B_2(bR_1 + dR_2) = A$. That is, $B_1\varphi_x + B_2\varphi_y = A$. If we multiply this equality by e^φ we obtain $B_1\varphi_x e^\varphi + B_2\varphi_y e^\varphi = Ae^\varphi$. Thus, $B_1(e^\varphi)_x + B_2(e^\varphi)_y = Ae^\varphi$. Therefore, the columns of e^φ are solutions for (66). $\square$

By using Proposition 3.1 solutions for the linear system with constant coefficients

$$a_1\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}=-b_1 y+b_1 z, \qquad (78)$$
$$-a_2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial t}=b_2 y-b_2 z,$$

can be constructed, as we see in the following example.

Example 3.1. Consider system (78). Let R_1, R_2, B_1, B_2, A be the matrices $R_1=I$, $B_2=I$,

$$R_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad B_1=\begin{pmatrix}a_1 & 0\\ 0 & -a_2\end{pmatrix},\quad A=\begin{pmatrix}-b_1 & b_1\\ b_2 & -b_2\end{pmatrix}.$$

Let $\mathbb{M}$ be the algebra spanned by R_1 and R_2. For $a_1+a_2\neq 0$ we define

$$\varphi(x,t)=\left(\frac{-b_1+b_2}{a_1+a_2}x+\frac{-a_1b_2-a_2b_1}{a_1+a_2}t\right)R_1+\left(\frac{b_1+b_2}{a_1+a_2}x+\frac{-a_1b_2+a_2b_1}{a_1+a_2}t\right)R_2.$$

Thus, $A=B_1\varphi_x+B_2\varphi_t$. By Proposition 3.1 the columns of

$$e^{\varphi(x,t)}=\begin{pmatrix}e^{h_1(x,t)}\cos h_2(x,t) & e^{h_1(x,t)}\sin h_2(x,t)\\ -e^{h_1(x,t)}\sin h_2(x,t) & e^{h_1(x,t)}\cos h_2(x,t)\end{pmatrix},$$

where

$$h_1(x,t)=\frac{-b_1+b_2}{a_1+a_2}x+\frac{-a_1b_2-a_2b_1}{a_1+a_2}t,$$
$$h_2(x,t)=\frac{b_1+b_2}{a_1+a_2}x+\frac{-a_1b_2+a_2b_1}{a_1+a_2}t,$$

are solutions for (78). Therefore,

$$y(x,t)=c_1e^{h_1(x,t)}\cos h_2(x,t)+c_2e^{h_1(x,t)}\sin h_2(x,t),$$
$$z(x,t)=-c_1e^{h_1(x,t)}\sin h_2(x,t)+c_2e^{h_1(x,t)}\cos h_2(x,t),$$

for all $c_1,c_2\in\mathbb{R}$, are solutions for (78).

4. Discussion and results

In this paper we introduce the $\varphi\mathbb{A}$-differentiability (pre-twisted differentiability), its generalized Cauchy–Riemann equations, the $\varphi\mathbb{A}$-line integral, the Cauchy integral theorem for the $\varphi\mathbb{A}$-line integral, and the $\varphi\mathbb{A}$-ODEs. Thus, a type of calculus on algebras and their corresponding differential equations has been introduced. The $\varphi\mathbb{A}$-differentiability is an extension of differentiability in the sense of Lorch [12].

When studying differential equations using pre-twisted calculus and their corresponding differential equations, the aim is to convert a differential equation problem into an algebraic problem. That is, given a differential equation or system of differential equations, one seeks to associate a system of algebraic equations whose solutions determine both $\mathbb{A}$ and φ in such a way that a function or a family of $\varphi\mathbb{A}$-differentiable functions determine solutions, see [16].

If a PDE system is equivalent to a set of generalized Cauchy–Riemann equations for the $\mathbb{A}$-differentiability (differentiability in the sense of Lorch), then the $\mathbb{A}$-differentiable functions constitute a complete solution for this system, see [21]. The $\varphi\mathbb{A}$-differentiability extends this result in a very impressive way. A criterion for $\varphi\mathbb{A}$-differentiability is given in Theorem 1.2. That is, if a given function satisfies a set of generalized Cauchy–Riemann equations for the $\varphi\mathbb{A}$-differentiability, then this function is $\varphi\mathbb{A}$-differentiable. Therefore, pre-twisted differentiability gives us a complete solution for these generalized Cauchy–Riemann equations.

The ODE system (27) and the $\varphi\mathbb{A}$-ODE (28) have the same solution set when φ' is invertible with respect to the $\mathbb{A}$-product and $f(t,x)\varphi'^{-1}$ is $(\varphi\mathbb{A},\mathbb{A})$-differentiable, see Lemma 2.1. If K in (31) is $\varphi\mathbb{A}$-differentiable, a non-singular solution w of (31) (non-singular solution w means that that the image of w is not contained in the singular set of $\mathbb{A}$) is also a solution of (33). Then, all the $\mathbb{A}$-products $\{aw\ :\ a\in\mathbb{A}\}$ of w and elements of $\mathbb{A}$ are solutions of (33). Thus, they are also solutions of (31). In this way a non-singular solution w of (31) could be used for constructing a fundamental set of solutions of this equation.

Consider the linear system

$$\frac{dw}{dt}=A(t)w, \qquad (79)$$

where $A: I\to M_n(\mathbb{R})$ is a continuous function and $A(t)w$ is the product of the matrix $A(t)$ by w, where $I\subset\mathbb{R}$ is an open interval. It is well-known that if $\{A(t)\ :\ t\in I\}$ is a commutative matrix family, that is, $A(t_1)A(t_2)=A(t_2)A(t_1)$ for all $t_1,t_2\in I$, then for $t_0\in I$ we have that

$$w(t)=e^{\int_{t_0}^{t}A(s)ds}c \qquad (80)$$

is a solution for system (79) with initial condition $w(t_0)=c\in\mathbb{R}^n$. The $\varphi\mathbb{A}$-ODEs defined in this paper could be used to solve linear and non-linear non-autonomous ODE systems, see Section 2.3. By Lemma 4.1 of [21] $ab=R(a)b$ for all $a,b\in\mathbb{A}$, where R if the first fundamental representation of $\mathbb{A}$. Thus, (31) is a linear ODE system of the type (79) for $A(t)=R(K(t)\varphi')$. Example 2.2 suggests considering inverse problems for the linear case. That is, given a linear system of the type (79), when there exists an algebra $\mathbb{A}$ and a differentiable function $\varphi(t)$ such that $A(t)=R(K(t)\varphi')$ for some $\varphi\mathbb{A}$-differentiable function K, as in Corollary 2.3. When you have an autonomous ODE system and it can be solved by Lorch differentiability, then it can also be solved by pre-twisted differentiability. The algebrizability (differentiability in the sense of Lorch) of planar autonomous ODE systems is already reached, see [17,19]. Example 2.3 contains a family of nonlinear ODE systems which are solved by using $\varphi\mathbb{A}$-differentiability. This example suggests considering inverse problems for these types of quadratic ODE systems. An attempt for solving planar non-autonomous ODE systems by using Lorch differentiability is made in [17]. This use of Lorch differentiability corresponds to pre-twisted differentiability for the particular case $\varphi(t)=te$ where $e\in\mathbb{A}$ is the identity for the product. It is worth mentioning that here we consider differential functions φ. Gâteaux differentiability is a weaker differentiability than the Lorch one, when the Lorch derivative of a function there exists then this derivative coincides with the Gâteaux derivative, see [5]. Therefore, Gâteaux differentiability considers larger families of differentiable functions than the Lorch case, but it does not include the cases that pre-twisted differentiability does for φ non-linear. In all cases of differential equations in which the Lorch differentiability can be used for the construction of solutions, the pre-twisted derivative can also be used, since the first one is a particular case of the second one. A weaker definition of a pre-twisted differentiability can be given, as we will do below. Let $f,\varphi:\mathbb{R}^k\to\mathbb{R}^n$ be functions having all the directional derivatives which we denote by d_hf and $d_h\varphi$, respectively, then f is said to be *$\varphi\mathbb{A}$-Gâteaux differentiable* with derivative $f'_{\varphi,G}:\mathbb{R}^k\to\mathbb{R}^n$ if $d_hf(x)=f'_{\varphi,G}(x)d_h\varphi(x)$. For the particular case where $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ is the identity the $\varphi\mathbb{A}$-Gâteaux differentiability coincides with the Gâteaux differentiability.

Pre-twisted differentiability is also used to solve Cauchy problems for PDEs of the form (2), see Section 3.1. This differentiability is a natural extension of the complex derivative. Just as the conjugate functions of holomorphic functions are solutions of Laplace's equation; for each equation of the family of PDEs (2) an algebra $\mathbb{A}$ and a linear function φ are found in such a way that the conjugate functions of the $\varphi\mathbb{A}$-differentiable functions determine the set of solutions of the considered equation.

In [32] we consider PDE systems of the type (66) and we try to algebrize them. This could help to build at least one particular solution of the system. Thus, for some linear PDE systems, knowing the solutions of the homogeneous system (the homogeneous system could be a set of generalized Cauchy–Riemann equations) allows us to give a complete solution for the system. We use the matrix exponential function $E=e^{\varphi}$ for the construction of solutions for linear PDE systems with constant coefficients of the type (66), see Proposition 3.1. In special cases we obtain a complete solution of this PDE systems, see Theorem 3.7. We show that $\varphi\mathbb{A}$-differentiable functions can also be used for the construction of solutions for quadratic PDE systems, see Theorem 3.8. We seek to generalize the idea of solving PDE systems using the columns of the exponential function. In [32] it is given a definition of algebrizability of (66) by using matrix algebras. We seek to algebraize other PDE systems by means of algebras, we work on this in [32].

The family of 2-dimensional algebras for which e_1 is the identity for the product is the 2-parameters algebras family consisting of the algebras $\mathbb{A}_1^2(p_1,p_2)$. Thus, the family of planar quadratic autonomous ODE systems which can be given in the form

$$\frac{dw}{dt}=\varphi' w^2 \tag{81}$$

for φ linear, is 4-dimensional. Let $p_1,p_2,\dots,p_9$ be constants satisfying the equalities

$$\begin{aligned}&p_7=p_1p_4+p_2p_6-p_2p_3-p_4^2,\\&p_8=p_3p_4-p_2p_5,\\&p_9=p_1p_5+p_3p_6-p_4p_5-p_3^2.\end{aligned}\tag{82}$$

Thus, the linear space $\mathbb{R}^3$ endowed with the following product

$\cdot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	$p_7e_1+p_1e_2+p_2e_3$	$p_8e_1+p_3e_2+p_4e_3$
e_3	e_3	$p_8e_1+p_3e_2+p_4e_3$	$p_9e_1+p_5e_2+p_6e_3$

(83)

is an algebra with identity $e=e_1$ which we denote by $\mathbb{A}_1^3(p_1,\dots,p_6)$. The family of 3-dimensional algebras for which e_1 is the identity for the product is the 6-parameters algebras family consisting of the algebras $\mathbb{A}_1^3(p_1,\dots,p_6)$. Thus, this is a 6-parameters algebras family. Then the family of three quadratic autonomous ODE systems that can be given by form (81) for φ linear, for $\mathbb{A}$ in this family, is 9-dimensional. In the case of 4-dimensional algebras with identity $e=e_1$ for the product we think that there are classes of algebras that depend on up to twelve parameters. Therefore, the family of systems of four quadratic autonomous ODEs that can be given by $dw/dt=\varphi' w^2$ for φ linear with respect to these classes of 12-parameter algebras is 16-dimensional. One can consider four dimensional autonomous ODE systems and investigate whether these can be written in the form (81) for φ linear. For example, the following

$$\begin{array}{lcl}\frac{dx_1}{dt}&=&b(x_1^2-y_1^2)-(b+c)(x_1x_2-y_1y_2)\\\frac{dy_1}{dt}&=&2bx_1y_1-(b+c)(x_1y_2+x_2y_1)\\\frac{dx_2}{dt}&=&a(x_2^2-y_2^2)-(a+c)(x_1x_2-y_1y_2)\\\frac{dy_2}{dt}&=&2ax_2y_2-(a+c)(x_1y_2+x_2y_1)\end{array}. \tag{84}$$

This family of ODE systems is associated with triangular billiards.

If we consider a quadratic ODE system of the form

$$\begin{array}{lcl}\frac{du}{dt}&=&a_1u^2+a_2uv+a_3v^2\\\frac{dv}{dt}&=&b_1u^2+b_2uv+b_3v^2,\end{array}\tag{85}$$

we can try algebrize it with respect to $\mathbb{A}=\mathbb{A}_1^3(p_1,\dots,p_6)$. So, we consider the ODE system

$$\frac{dw}{dt}=(a_1w_1^2+a_2w_1w_2+a_3w_2^2,b_1w_1^2+b_2w_1w_2+b_3w_2^2,0).\tag{86}$$

Thus, we would like to write them by

$$\frac{dw}{dt}=(h,k,l)w^2,\tag{87}$$

with respect to $\mathbb{A}$. Thus,

$$\begin{array}{lcl}\frac{dw_1}{dt}&=&(-lp_2p_3^2-hp_2p_3-kp_1p_2p_3-hp_4^2-kp_1p_4^2+kp_1^2p_4\\&&+hp_1p_4+lp_1p_3p_4)w_1^2\\&+&(kp_2p_3p_4-kp_2^2p_5-lp_2p_4p_5+hp_2p_6+kp_1p_2p_6+lp_2p_3p_6)w_1^2\\&+&(-2kp_2p_3^2+2hp_3p_4+2kp_1p_3p_4-2hp_2p_5-2lp_2p_3p_5\\&&-2lp_4^2p_5)w_1w_2\\&+&(2lp_1p_4p_5-2kp_2p_4p_5+2kp_2p_3p_6+2lp_3p_4p_6)w_1w_2\\&+&(-hp_3^2-lp_2p_5^2+hp_1p_5-kp_2p_3p_5-kp_4^2p_5-hp_4p_5\\&&+kp_1p_4p_5+lp_3p_4p_5)w_2^2\\&+&(lp_3p_6^2-lp_3^2p_6+hp_3p_6+kp_3p_4p_6+lp_1p_5p_6-lp_4p_5p_6)w_2^2,\\\frac{dw_2}{dt}&=&(kp_1^2+hp_1+lp_1p_3-kp_4^2+kp_1p_4+lp_2p_5+kp_2p_6)w_1^2\\&+&(2lp_3^2+2hp_3+2kp_1p_3+4kp_3p_4-2kp_2p_5+2lp_4p_5)w_1w_2\\&+&(-kp_3^2+hp_5+2kp_1p_5+lp_3p_5-kp_4p_5+2kp_3p_6+lp_5p_6)w_2^2,\\\frac{dw_3}{dt}&=&(hp_2+kp_1p_2-lp_2p_3-lp_4^2+2lp_1p_4+kp_2p_4+2lp_2p_6)w_1^2\\&+&(2kp_2p_3+2kp_4^2+2hp_4+4lp_3p_4-2lp_2p_5+2lp_4p_6)w_1w_2\\&+&(-lp_3^2+lp_1p_5+kp_2p_5+lp_6^2+hp_6+lp_3p_6+kp_4p_6)w_2^2.\end{array}\tag{88}$$

We would like to have $(h,k,l,p_1,\dots,p_6)$ such that systems (86) and (88) coincide. Thus, $(h,k,l,p_1,\dots,p_6)$ must be a solution for the four order algebraic system in the variables x_i for $i=1,\dots,9$

$$\begin{array}{lcl}-x_3x_5x_6^2-x_1x_5x_6-x_2x_4x_5x_6-x_1x_7^2-x_2x_4x_7^2+x_2x_4^2x_7&&\\+x_1x_4x_7+x_3x_4x_6x_7&&\\+x_2x_5x_6x_7-x_2x_5^2x_8-x_3x_5x_7x_8+x_1x_5x_9+x_2x_4x_5x_9&&\\+x_3x_5x_6x_9&=&a_1\\-2x_2x_5x_6^2+2x_1x_6x_7+2x_2x_4x_6x_7-2x_1x_5x_8-2x_3x_5x_6x_8&&\\-2x_3x_7^2x_8&&\\+2x_3x_4x_7x_8-2x_2x_5x_7x_8+2x_2x_5x_6x_9&&\\+2x_3x_6x_7x_9&=&a_2\\-x_1x_6^2-x_3x_5x_8^2+x_1x_4x_8-x_2x_5x_6x_8-x_2x_7^2x_8-x_1x_7x_8&&\\+x_2x_4x_7x_8+x_3x_6x_7x_8&&\\+x_3x_6x_9^2-x_3x_6^2x_9+x_1x_6x_9+x_2x_6x_7x_9+x_3x_4x_8x_9&&\\-x_3x_7x_8x_9&=&a_3\\x_2x_4^2+x_1x_4+x_3x_4x_6-x_2x_7^2+x_2x_4x_7+x_3x_5x_8&&\\+x_2x_5x_9&=&b_1\\2x_3x_6^2+2x_1x_6+2x_2x_4x_6+4x_2x_6x_7-2x_2x_5x_8&&\\+2x_3x_7x_8&=&b_2\\-x_2x_6^2+x_1x_8+2x_2x_4x_8+x_3x_6x_8-x_2x_7x_8+2x_2x_6x_9&&\\+x_3x_8x_9&=&b_3\\x_1x_5+x_2x_4x_5-x_3x_5x_6-x_3x_7^2+2x_3x_4x_7+x_2x_5x_7&&\\+2x_3x_5x_9&=&0\\2x_2x_5x_6+2x_2x_7^2+2x_1x_7+4x_3x_6x_7-2x_3x_5x_8&&\\+2x_3x_7x_9&=&0\\-x_3x_6^2+x_3x_4x_8+x_2x_5x_8+x_3x_9^2+x_1x_9+x_3x_6x_9&&\\+x_2x_7x_9&=&0.\end{array}\tag{89}$$

If $(h,k,l,p_1,\dots,p_6)$ is a solution for the algebraic system (89), we obtain system (86) and

$$w(t)=\frac{-e}{\varphi(t)+C},\tag{90}$$

where $\varphi(t)=t(h,k,l)$ and $C=(c_1,c_2,0)$, is a solution.

If $(h,k,l,p_1,\dots,p_6)$ is a solution for the algebraic system (89) for

$$(a_1,a_2,a_3,b_1,b_2,b_3)=(b,-(b+c),0,0,-(a+c),a),$$

then ODE system (88) become

$$\begin{array}{lcl}\frac{dw_1}{dt}&=&bw_1^2-(b+c)w_1w_2\\\frac{dw_2}{dt}&=&-(a+c)w_1w_2+aw_2^2.\end{array}\tag{91}$$

If we take this system over the complex field $\mathbb{C}$ ($w_j=x_j+iy_j$ for $j=1,2$) it corresponds to the system (84) written in complex variables. Thus, a solution for ODE system (91) is given by (90).

Declaration of competing interest

The authors declare that there is no conflict of interests. The issues discussed in this paper do not have any secondary interest for any of the authors.

Data availability

No data was used for the research described in the article.

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