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ORIGINAL ARTICLE

Improved cosine similarity and distance measures-based TOPSIS method for cubic Fermatean fuzzy sets

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Abstract Similarity and distance measures play important roles in fuzzy environments, helping to quantify the degree of similarity or concepts that may not have clear limits. They are used in various fields, including fuzzy logic, fuzzy clustering, and fuzzy decision-making. The cubic Fermatean fuzzy set (CFFS), which is a type of fuzzy set (FS), is highly favoured as an extension for expressing uncertainty through degrees of membership (η) and non-membership (υ). This article introduces novel measures for cosine similarity and distance in CFFSs. These measures are designed to improve the accuracy and efficiency of similarity and distance calculations in CFFSs. Also, a novel method is introduced for developing alternate similarity measures for CFFSs utilizing the proposed similarity measures that adhere to the similarity measures axiom. In addition, the connection between similarity and distance measures is utilized to construct a cosine distance metric for CFFSs. This newly suggested cosine similarity measure can not only provide solutions to decision-making problems from a geometric perspective but also from an algebraic point of view. To conclude, a case study is presented to showcase the practicality and effectiveness of the proposed approach, followed by a comparison of the outcomes of the suggested technique with some existing methodologies. This analysis helps to validate the proposed method and demonstrates its potential for outperforming other available approaches in terms of efficiency and accuracy.

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1. Introduction

Multi-criteria decision-making (MCDM) is a decision-making process that considers multiple criteria simultaneously to reach a preferred alternative among several options. MCDM is used in various fields such as engineering, management, economics, environmental sciences, and social sciences. The main advantage of MCDM is that it enables decision-makers to take into account multiple factors and criteria that may have different weights and importance in the decision-making process. This allows for a more comprehensive evaluation of the options, leading to better-informed decisions. In certain practical scenarios, decision-makers are unable to provide precise numerical values for their ratings of the alternatives. To fill this gap Zedah introduced fuzzy sets [1] which provide a powerful and flexible framework for modelling uncertainty and imprecision in various fields. Their ability to represent degrees of membership and their extensions, including fuzzy numbers and fuzzy logic, make them valuable tools for solving complex problems that involve uncertainty and imprecision. Atanassov [2] extended the idea of Zedah's fuzzy sets and proposed intuitionistic fuzzy sets (IFS). Several researchers have employed IFS in various fields such as decision-making [3], pattern recognition [4], control systems [5], and image processing [6,7]. To tackle uncertainty in decision-making, Yager [8] introduced the concept of Pythagorean fuzzy sets (PFSs). PFSs provide a framework for representing fuzzy sets that incorporate both membership degrees and non-membership degrees, allowing for a more comprehensive assessment of alternatives. Pythagorean Fuzzy Sets (PFSs) have been applied in diverse areas, such as MCDM [9], image processing for pattern recognition and segmentation [10], intelligent design for controlling complex and uncertain systems [11], classification and clustering for data mining [12,13], and managing data that are uncertain or imprecise [14,15]. Table 1 provides explanations for the abbreviations used in this article.

Table 1 List of abbreviations.

Abbreviations	Explanations	Abbreviations	Explanations
MCDM	Multi-criteria decision-making	PFS	Pythagorean fuzzy set
CFFS	Cubic Fermatean fuzzy set	FS	Fuzzy set
CPFS	Cubic Pythagorean fuzzy set	FFS	Fermatean fuzzy set
CIFS	Cubic intuitionistic fuzzy set	IVFFS	Interval-valued Fermatean fuzzy set
IVPFS	Interval-valued Pythagorean fuzzy set	CS	Cubic set
IVIFS	Interval-valued intuitionistic fuzzy set	CFFN	Cubic Fermatean fuzzy number
IFS	Intuitionistic fuzzy set	CIFN	Cubic intuitionistic fuzzy number

The available literature indicates that the primary emphasis of previous research has been on the FSs, interval FSs (IVFSs), IFSs, PFSs, and the practical applications of these concepts. Subsequently, Jun et al. [16] combined interval-valued fuzzy numbers and fuzzy numbers, proposed the notion of cubic sets (CSs) and established logical operations for these sets. Under this set, Khan et al. [17] proposed cubic aggregation operators, while Mahmood et al. [18] introduced the concept of cubic hesitant fuzzy sets and related aggregation operators to solve decision-making problems. Garg and Kaur [19] proposed cubic IFSs (CIFSs) by integrating interval-valued IFSs (IVIFSs). Kaur and Garg [20] proposed Bonferroni mean and weighted Bonferroni mean averaging operators between cubic intuitionistic fuzzy numbers. Chinnadurai and colleagues [21] introduced the concept of complex cubic intuitionistic fuzzy sets (CCIFS), which are a combination of complex cubic membership values and complex cubic non-membership values, and defined several related operations for these sets. Amin et al. [22] proposed generalized cubic PFSs (CPFSs) and their operational laws. Rahim et al. [23] presented Bonferroni mean aggregation operators and their application in MCDM problems.

The application of the theory of cubic intuitionistic fuzzy numbers is limited to cases where the condition $\eta^U + \upsilon^U \leqslant 1$ holds. Similarly, Pythagorean fuzzy numbers can only be utilized when the condition $(\eta^U)^2 + (\upsilon^U)^2 \leqslant 1$ is met. However, in certain real-world situations, these conditions may not be satisfied, thereby posing a challenge to decision-makers. For instance, when a decision-maker rates an alternative as (0.8, 0.7), it cannot be accommodated by either intuitionistic fuzzy sets (IFS) or Pythagorean fuzzy sets (PFS). To address this gap, Rong et al. [24] proposed Fermatean fuzzy sets such that $(\eta^U)^3 + (\upsilon^U)^3 \leqslant 1$, provide a solution by relaxing the limitations of IFS and PFS. This new fuzzy set theory allows decision-makers to handle complex and uncertain information that falls outside the boundaries of the traditional fuzzy sets. Fermatean fuzzy sets have attracted the attention of many scholars, who have been actively involved in their development and application in different fields. As a result, various operational laws and aggregation operators have been proposed to better represent complex and uncertain information see [25,26]. Table 2 contains the list of notations utilized in this article.

1.1. Overview of similarity and distance measures

Almost every methodological approach mostly depend on the idea of similarity. For example, the study of homothetic transformation and symmetry along with related disciplines like trigonometry and geometric approaches are used to evaluate similarities. Further, Similarity measures are generalized to the fuzzy environment, which is applied to business, health, and climatology domains. Unsurprisingly, similarity has been essential in psychological sciences and interpretations. For example, many investigations require participants to make direct or indirect judgements regarding the similarity of two objects. Several experimental processes are used in these studies. In the field of data science, similarity measures are employed to evaluate the extent of similarity between different data samples. The goal is to determine the level of relatedness between these samples. On the contrary, dissimilarity measures are utilized to assess the degree of difference or uniqueness

Table 2 List of notations.

η^U	The upper limit of the membership function	F	Fermatean fuzzy set	A^-	Negative ideals
υ^U	The upper limit of the non-membership function	A	Cubic Fermatean fuzzy set	A^+	Positive ideals
η	member membership function	Sc	Score function	ω	Wight vector
υ	non-membership function	Ac	Accuracy function	$\widetilde{CS}$	Similarity measure
G	Any non-empty finite set	π	Fermatean fuzzy index	H	Criteria

between data objects. Both of these measures are frequently utilized in clustering, a method for grouping data samples that are similar to each other into one cluster. It's important to note that the concept of similarity is subjective and can vary depending on the context and purpose of the analysis. For instance, when evaluating the similarity of various vegetables, factors such as taste, size, and colour can be taken into consideration to determine how similar they are to each other.

In fuzzy set theory, the concept of similarity measures is of utmost importance. It is widely used in various domains such as pattern recognition, medical diagnostics, and others. Several similarity measures have been extensively studied in the context of FSs, IFSs, and PFSs. These measures play a critical role in determining the level of similarity between different elements, which is essential in many real-world applications. For example, Liao et al. [27] conducted a study and developed various distance and similarity measures for hesitant fuzzy linguistic term sets. In their research, they explored different methods to determine the similarity and distance between hesitant fuzzy linguistic terms, to improve the accuracy and efficiency of pattern recognition and other related tasks. Xuecheng [28] systematically presented the axiom definitions of entropy, distance measure, and similarity measure of fuzzy sets, and explained the fundamental connections between these measures. Xu and Shen [29] introduced the similarity measures of Fermatean fuzzy sets. The paper defined both the regular and weighted similarity measures for Fermatean fuzzy sets on both discrete and continuous universes. Wei and Wei [30] introduced ten distinct similarity measures between PFSs that utilize the cosine function. The measures take into consideration the membership, non-membership, and hesitation degree within PFSs. Ye [31] introduced cosine similarity and weighted cosine similarity measures between IFSs, based on the cosine similarity measure for FSs. Zeng and Li [32] presented the concepts of entropy and similarity measure for interval-valued FSs and extensively examined their relationship. Zhou et al. [33] introduced a new similarity measure combining the cosine similarity measure for intuitionistic fuzzy sets and a generalized ordered weighted averaging operator. Additional discussion on the application of various similarity measures defined in different environments can be found in sources [34–37].

Based on the analysis presented earlier, there hasn't been any research that has specifically looked into cosine similarity and distance in the cubic Fermatean fuzzy environment. Although the CIFS and CPFS are both included in this environment, the CPFSs are better equipped to handle greater levels of vagueness and uncertainty. To adequately manage the intricate uncertainties involved in decision-making and human cognition, it's important to define the similarty and distance measures in CFFS context.

1.2. Motivation and originality

Similarity measures and distance measures play a significant role in evaluating and comparing cubic Fermatean fuzzy numbers (CFFNs). However, the selection of appropriate similarity or distance measures primarily depends on the specific problem being addressed and the context. Currently, there is a lack of research in the literature for comparing two CFFNs. Nevertheless, the flexible nature of CFFNs motivated researchers to propose a novel cosine similarity and the distance measure for CFFNs in this study. The proposed measures are developed based on the cosine similarity measure and the Euclidean distance measure, respectively. The cosine similarity measure is used to evaluate the degree of similarity between two CFFNs. Similarly, the Euclidean distance measure is utilized to measure the distance between two CFFNs. An in-depth analysis is conducted to examine the characteristics of the proposed cosine similarity and distance measures. By utilizing these measures, a novel multi-criteria decision-making approach is introduced by integrating them with the Technique for Order Performance by Similarity to Ideal Solution (TOPSIS) method.

The article is organized as follows. In Section 2, the necessary background information for the study is provided. In Section 3, two new cosine similarity measures, cosine similarity and weighted cosine similarity, are introduced and their properties are discussed. Section 4 focuses on the MCDM algorithm using cosine similarities and the TOPSIS method. In the fifth section, the application of the cosine similarities in the field of infectious diseases is presented and a medical decision-making model is introduced to demonstrate the ease of use and effectiveness of the developed cosine similarities in real-life decision-making problems. The results of the case study illustrate that the CFFS framework-based cosine similarities can effectively handle decision-making problems with multiple objectives.

2. Preliminaries

The following section will provide the essential background information that will be utilized in the study.

2.1. Fermatean fuzzy sets

Definition 1. [38] Given a non-empty, finite set G, a Fermatean fuzzy set (FFS) associated with an element $g \in G$ is defined as the follows:

$$F = \{g, \langle \eta_F(g), \upsilon_F(g) \rangle | g \in G\} \quad (1)$$

where $\eta_F(g)$ represent the membership degree (MD) and $\upsilon_F(g)$ represent the non-membership degree (NMD) of an element $g \in G$, such that $0 \preccurlyeq \eta_F(g), \upsilon_F(g) \preccurlyeq 1$ satisfying the condition $\eta_F^3(g) + \upsilon_F^3(g) \preccurlyeq 1$ for every element $g \in G$. For the sake of simplicity $(\eta_F(g), \upsilon_F(g))$ is referred to as a FFN, which can be represented as $\beta = (\eta_\beta, \upsilon_\beta)$, satisfying the condition $\eta_\beta^3 + \upsilon_\beta^3 \preccurlyeq 1$. The indeterminacy degree of g to F is defined as:

$$\pi_F = \sqrt{1 - (\eta_F(g))^3 + (\upsilon_F(g))^3} \tag{2}$$

Definition 2. [38] Let $\beta = (\eta_\beta, \upsilon_\beta)$ be a FFN, then the score function over G is defined as:

$$\mathrm{Sc}(\beta) = (\eta_\beta)^3 - (\upsilon_\beta)^3 \tag{3}$$

where $-1 \leq \mathrm{Sc}(\beta) \leq 1$. The accuracy function of β is defined as:

$$\mathrm{Ac}(\beta) = (\eta_\beta)^3 + (\upsilon_\beta)^3 \tag{4}$$

$0 \leq \mathrm{Ac}(\beta) \leq 1$

Let β_1 and β_2 be two FFNs, then the following relationships hold:

1. If $\mathrm{Sc}(\beta_1) \prec \mathrm{Sc}(\beta_2)$ then $\beta_1 \prec \beta_2$,
2. If $\mathrm{Sc}(\beta_1) \succ \mathrm{Sc}(\beta_2)$ then $\beta_1 \succ \beta_2$,
3. If $\mathrm{Sc}(\beta_1) = \mathrm{Sc}(\beta_2)$ then
   i. If $\mathrm{Ac}(\beta_1) = \mathrm{Ac}(\beta_2)$ then $\beta_1 = \beta_2$,
   ii. If $\mathrm{Ac}(\beta_1) \prec \mathrm{Ac}(\beta_2)$ then $\beta_1 \prec \beta_2$,
   iii. If $\mathrm{Ac}(\beta_1) \succ \mathrm{Ac}(\beta_2)$ then $\beta_1 \succ \beta_2$.

2.2. Cubic sets

Jun et al. [16] presented the idea of cubic sets (CSs). In this section, some basic definitions related to CSs are discussed in detail.

Definition 3. [16] Let G be a universal set. A cubic set (CS) over an element $g \in G$ is defined as:

$$C = \left\{g, \widetilde{A}_C(g), \mu_C(g) | g \in G\right\} \tag{5}$$

where $\widetilde{\boldsymbol{A}}_{\boldsymbol{C}}(\boldsymbol{g}) = [\boldsymbol{\eta}_{\boldsymbol{C}}^{\boldsymbol{L}}(\boldsymbol{g}), \boldsymbol{\eta}_{\boldsymbol{C}}^{\boldsymbol{U}}(\boldsymbol{g})]$ is an IVFS and $\boldsymbol{\mu}_{\boldsymbol{C}}(\boldsymbol{g})$ is a FS in set $\boldsymbol{G}$ such that $0 \preccurlyeq \boldsymbol{\eta}_{\boldsymbol{C}}^{\boldsymbol{L}}(\boldsymbol{g}) \preccurlyeq \boldsymbol{\eta}_{\boldsymbol{C}}^{\boldsymbol{U}}(\boldsymbol{g}) \preccurlyeq 1$ and $0 \preccurlyeq \boldsymbol{\mu}_{\boldsymbol{C}}(\boldsymbol{g}) \preccurlyeq 1$.

Definition 4. [16] A cubic set $C = \left(\widetilde{A}_C(g), \mu_C(g)\right)$ is said to be an internal cubic set if $\mu_C(g) \in [\eta_C^L(g), \eta_C^U(g)]$.

Definition 5. [16] A cubic set $C = \left(\widetilde{A}_C(g), \mu_C(g)\right)$ is said to be an external cubic set if $\mu_C(g) \notin [\eta_C^L(g), \eta_C^U(g)]$.

2.3. Cubic FFSs and their existing operations

Definition 5. A cubic Fermatean fuzzy set (CFFS) over an element g that belongs to a non-empty and finite set G is described as follows:

$$A = \{g, B(g), \psi(g) | g \in G\} \tag{6}$$

where $B(g) = \{g, \langle [\eta_A^L(g), \eta_A^U(g)], [\upsilon_A^L(g), \upsilon_A^U(g)] \rangle\}$ represents interval-valued Fermatean fuzzy set while $\psi(x) = \{g, \eta_A(g), \upsilon_A(g)\}$ represents FFS for all $g \in G$ such that $0 \preccurlyeq \eta_A^L(g) \preccurlyeq \upsilon_A^U(g) \preccurlyeq 1$, $0 \preccurlyeq \upsilon_A^L(g) \preccurlyeq \upsilon_A^U(g) \preccurlyeq 1$ and $0 \preccurlyeq (\eta_A^U(g))^3 + (\upsilon_A^U(g))^3 \preccurlyeq 1$. Also, $0 \preccurlyeq \eta_A(g), \upsilon_A(g) \preccurlyeq 1$ and $0 \preccurlyeq (\eta_A(g))^3 + (\upsilon_A(g))^3 \preccurlyeq 1$. To keep it simple, the pair $\alpha = (A, \psi)$, where $A = \langle [\eta_A^L, \eta_A^U], [\upsilon_A^L, \upsilon_A^U] \rangle$ and $\psi = \langle \eta_A, \upsilon_A \rangle$ and called as cubic Fermatean fuzzy number (CFFN). The conditions for CFFN can be summarized as follows:

1. η_A^L, $\eta_A^U \in [0,1]$, υ_A^L, $\upsilon_A^U \in [0,1]$, and $\eta_A, \upsilon_A \in [0,1]$.
2. $0 \preccurlyeq (\eta_A^U)^3 + (\upsilon_A^U)^3 \preccurlyeq 1$ and $(\eta_A)^2 + (\upsilon_A)^2 \preccurlyeq 1$.

Definition 6. Let $\alpha = (\langle [\eta_A^L, \eta_A^U], [\upsilon_A^L, \upsilon_A^U] \rangle, \langle \eta_A, \upsilon_A \rangle)$, $\alpha_i = \left(\left\langle \left[\eta_{A_i}^L, \eta_{A_i}^U\right], \left[\upsilon_{A_i}^L, \upsilon_{A_i}^U\right] \right\rangle, \langle \eta_{A_i}, \upsilon_{A_i} \rangle\right)$ $(i = 1,2)$ be the collections of cubic Fermatean fuzzy numbers (CFFNs), and $\omega \succ 0$ be a real number then

1. $$\alpha_1 \oplus \alpha_2 = \left(\left\langle \begin{bmatrix} \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - (\eta_{A_i}^L)^3\right)}, \\ \sqrt[3]{1 - \prod_{i=1}^2 (1 - (\eta_{A_i}^U)^p)} \end{bmatrix}, \begin{bmatrix} \prod_{i=1}^2 \upsilon_{A_i}^L, \\ \prod_i^2 \upsilon_{A_i}^U \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \prod_{i=1}^2 \eta_{A_i}, \\ \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - (\upsilon_{A_i})^3\right)} \end{matrix} \right\rangle\right);$$
2. $$\alpha_1 \otimes \alpha_2 = \left(\left\langle \begin{bmatrix} \prod_{i=1}^2 \eta_{A_i}^L, \\ \prod_{i=1}^2 \eta_{A_i}^U \end{bmatrix}, \begin{bmatrix} \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - (\upsilon_{A_i}^L)^3\right)}, \\ \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - (\upsilon_{A_i}^U)^3\right)} \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - (\eta_{A_i})^3\right)}, \\ \prod_{i=1}^2 \upsilon_{A_i} \end{matrix} \right\rangle\right);$$
3. $$\omega\alpha = \left(\left\langle \begin{bmatrix} \sqrt[3]{1 - \left(1 - (\eta_{A_i}^L)^3\right)^\omega}, \\ \sqrt[3]{1 - \left(1 - (\eta_{A_i}^U)^3\right)^\omega} \end{bmatrix}, \begin{bmatrix} (\upsilon_{A_i}^L)^\omega, \\ (\upsilon_{A_i}^U)^\omega \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \eta_{A_i}^\omega, \\ \sqrt[3]{1 - (1 - \upsilon_{A_i}^3)^\omega} \end{matrix} \right\rangle\right);$$
4. $$\alpha^\omega = \left(\left\langle \begin{bmatrix} (\eta_{A_i}^L)^\omega, \\ (\eta_{A_i}^U)^\omega \end{bmatrix}, \begin{bmatrix} \sqrt[3]{1 - \left(1 - (\upsilon_{A_i}^L)^3\right)^\omega}, \\ \sqrt[3]{1 - \left(1 - (\upsilon_{A_i}^U)^3\right)^\omega} \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \sqrt[3]{1 - (1 - \eta_{A_i}^3)^\omega}, \\ \upsilon_{A_i}^\omega \end{matrix} \right\rangle\right).$$

Definition 7. [39] Let $\alpha = (\langle [\eta_A^L, \eta_A^U], [\upsilon_A^L, \upsilon_A^U] \rangle, \langle \eta_A, \upsilon_A \rangle)$ be a CFFN, then the score function under R-order is defined as:

$$Sc(\alpha) = \frac{(\eta_A^L)^3 + (\eta_A^U)^3 - (\upsilon_A^L)^3 - (\upsilon_A^U)^3}{2} + (\upsilon_A)^3 - (\eta_A)^3 \tag{7}$$

and for P-order the score function is defined as:

$$Sc(\alpha) = \frac{(\eta_A^L)^3 + (\eta_A^U)^3 - (\upsilon_A^L)^3 - (\upsilon_A^U)^3}{2} + (\eta_A)^3 - (\upsilon_A)^3 \tag{8}$$

$-1 \le Sc(\alpha) \le 1$

Definition 8. [39] Let $\alpha = \left(\left\langle [\eta_A^L, \eta_A^U], [\upsilon_A^L, \upsilon_A^U] \right\rangle, \langle \eta_A, \upsilon_A \rangle\right)$ be a CFFN, then the accuracy function under R-order (P-order) is defined as:

$$Ac(\alpha) = \frac{(\eta_A^L)^3 + (\eta_A^U)^3 + (\upsilon_A^L)^3 + (\upsilon_A^U)^3}{2} + (\eta_A)^3 + (\upsilon_A)^3 \tag{9}$$

3. Innovative techniques and measures for operating cubic FFNs

Definition 9. For a family of CFFSs $\{\alpha_i, i \in \Gamma\}$, then

1. (P-union): $\bigcup_{i\in\Gamma}^{P} \alpha_i = \left(\left\langle \begin{bmatrix} \max_{i\in\Gamma}(\eta_{A_i}^L), \\ \max_{i\in\Gamma}(\eta_{A_i}^U) \end{bmatrix}, \begin{bmatrix} \min_{i\in\Gamma}(\upsilon_{A_i}^L), \\ \min_{i\in\Gamma}(\upsilon_{A_i}^U) \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \max_{i\in\Gamma}\eta_{A_i}, \\ \min_{i\in\Gamma}\upsilon_{A_i} \end{matrix} \right\rangle\right)$;
2. (P-intersection): $\bigcap_{i\in\Gamma}^{R} \alpha_i = \left(\left\langle \begin{bmatrix} \min_{i\in\Gamma}(\eta_{A_i}^L), \\ \min_{i\in\Gamma}(\eta_{A_i}^U) \end{bmatrix}, \begin{bmatrix} \max_{i\in\Gamma}(\upsilon_{A_i}^L), \\ \max_{i\in\Gamma}(\upsilon_{A_i}^U) \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \min_{i\in\Gamma}\eta_{A_i}, \\ \max_{i\in\Gamma}\upsilon_{A_i} \end{matrix} \right\rangle\right)$;
3. (R-union): $\bigcup_{i\in\Gamma}^{R} \alpha_i = \left(\left\langle \begin{bmatrix} \max_{i\in\Gamma}(\eta_{A_i}^L), \\ \max_{i\in\Gamma}(\eta_{A_i}^U) \end{bmatrix}, \begin{bmatrix} \min_{i\in\Gamma}(\upsilon_{A_i}^L), \\ \min_{i\in\Gamma}(\upsilon_{A_i}^U) \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \min_{i\in\Gamma}\eta_{A_i}, \\ \max_{i\in\Gamma}\upsilon_{A_i} \end{matrix} \right\rangle\right)$;
4. (R-intersection): $\bigcup_{i\in\Gamma}^{P} \alpha_i = \left(\left\langle \begin{bmatrix} \max_{i\in\Gamma}(\eta_{A_i}^L), \\ \max_{i\in\Gamma}(\eta_{A_i}^U) \end{bmatrix}, \begin{bmatrix} \min_{i\in\Gamma}(\upsilon_{A_i}^L), \\ \min_{i\in\Gamma}(\upsilon_{A_i}^U) \end{bmatrix} \right\rangle, \left\langle \begin{matrix} \min_{i\in\Gamma}\eta_{A_i}, \\ \max_{i\in\Gamma}\upsilon_{A_i} \end{matrix} \right\rangle\right)$.

Definition 10. Let $\alpha_i = \left(\left\langle \left[\eta_{A_i}^L, \eta_{A_i}^U\right], \left[\upsilon_{A_i}^L, \upsilon_{A_i}^U\right] \right\rangle, \langle \eta_{A_i}, \upsilon_{A_i} \rangle\right)$ $(i = 1, 2)$ be the collections of CFFNs. Then $\pi_{A_i}(g) = \left(\left[\pi_{A_i}^L(g), \pi_{A_i}^U(g)\right], \pi_{A_i}(g)\right)$ is said to be cubic Fermatean fuzzy index of an element $g \in G$.

Where

$$\pi_{A_i}^L(g) = \sqrt[3]{1 - \left(\vartheta_{A_i}^U\right)^3 - \left(\psi_{A_i}^U\right)^3} \tag{10}$$

$$\pi_{A_i}^U(g) = \sqrt[3]{1 - \left(\vartheta_{A_i}^L\right)^3 - \left(\psi_{A_i}^L\right)^3} \tag{11}$$

$$\pi_{A_i}(g) = \sqrt[l]{1 - (\vartheta_{A_i})^3 - (\psi_{A_i})^3} \tag{12}$$

Example 1. Let $\alpha = (\langle [0.5, 0.6], [0.4, 0.5] \rangle, \langle 0.6, 0.7 \rangle)$ be a CFFN Then

$$\pi_A^L = \sqrt[3]{1 - (0.6)^3 - (0.5)^3} = 0.8702$$

$$\pi_{A_i}^U(g) = \sqrt[3]{1 - (0.5)^3 - (0.4)^3} = 0.9326$$

$$\pi_{A_i}(g) = \sqrt[3]{1 - (0.6)^3 - (0.7)^3} = 0.7612$$

3.1. Cosine similarity measure between CFFNs

The cosine similarity ($\widetilde{CS}$) measure is determined by taking the inner product of two vectors and dividing the result by the product of the lengths of these vectors. This is a well-established method for calculating $\widetilde{CS}$.

Definition 11. Let G be a non-empty finite set. For any two CFFNs $\vartheta = \left(\left\langle \left[\eta_{\vartheta_i}^L, \eta_{\vartheta_i}^U\right], \left[\upsilon_{\vartheta_i}^L, \upsilon_{\vartheta_i}^U\right] \right\rangle, \langle \eta_{\vartheta_i}, \upsilon_{\vartheta_i} \rangle\right)$ and $\delta = \left(\left\langle \left[\eta_{\delta_i}^L, \eta_{\delta_i}^U\right], \left[\upsilon_{\delta_i}^L, \upsilon_{\delta_i}^U\right] \right\rangle, \langle \eta_{\delta_i}, \upsilon_{\delta_i} \rangle\right)$ the $\widetilde{CS}$ between ϑ and δ is defined as:

$$\mathscr{C}_{CFF}(\alpha, \beta) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{A_i^L + A_i^U + B_i^L + B_i^U + C_i^L + C_i^U + A_i + B_i + C_i}{\sqrt[3]{a_i + b_i + c_i + d_i}\sqrt[3]{\widetilde{a}_i + \widetilde{b}_i + \widetilde{c}_i + \widetilde{d}_i}}\right) \tag{13}$$

Where $A_i^L = \left(\eta_{\vartheta_i}^L\right)^3\left(\eta_{\delta_i}^L\right)^3$, $A_i^U = \left(\eta_{\vartheta_i}^U\right)^3\left(\eta_{\delta_i}^U\right)^3$, $B_i^L = \left(\upsilon_{\vartheta_i}^L\right)^3\left(\upsilon_{\delta_i}^L\right)^3$, $B_i^U = \left(\upsilon_{\vartheta_i}^U\right)^3\left(\upsilon_{\delta_i}^U\right)^3$, $C_i^L = \left(\pi_{\vartheta_i}^L\right)^3\left(\pi_{\delta_i}^L\right)^3$, $C_i^L = \left(\pi_{\vartheta_i}^L\right)^3\left(\pi_{\delta_i}^L\right)^3$, $C_i^U = \left(\pi_{\vartheta_i}^U\right)^3\left(\pi_{\delta_i}^U\right)^3$, $A_i = (\eta_{\vartheta_i})^3(\eta_{\delta_i})^3$, $B_i = (\upsilon_{\vartheta_i})^3(\upsilon_{\delta_i})^3$, $C_i = (\pi_{\vartheta_i})^3(\pi_{\delta_i})^3$, $a_i = \left(\eta_{\vartheta_i}^L\right)^3 + \left(\eta_{\vartheta_i}^U\right)^3$, $b_i = \left(\upsilon_{\vartheta_i}^L\right)^3 + \left(\upsilon_{\vartheta_i}^U\right)^3$, $c_i = \left(\pi_{\vartheta_i}^L\right)^3 + \left(\pi_{\vartheta_i}^U\right)^3$, $d_i = (\eta_{\vartheta_i})^3 + (\upsilon_{\vartheta_i})^3 + (\pi_{\vartheta_i})^3$, $\widetilde{a}_i = \left(\eta_{\delta_i}^L\right)^3 + \left(\eta_{\delta_i}^U\right)^3$, $\widetilde{b}_i = \left(\upsilon_{\delta_i}^L\right)^3 + \left(\upsilon_{\delta_i}^U\right)^3$, $\widetilde{c}_i = \left(\pi_{\delta_i}^L\right)^3 + \left(\pi_{\delta_i}^U\right)^3$, and $\widetilde{d}_i = (\eta_{\delta_i})^3 + (\upsilon_{\delta_i})^3 + (\pi_{\delta_i})^3$.

Theorem 1. Let $\alpha_i = \left(\left\langle \left[\eta_{\alpha_i}^L, \eta_{\alpha_i}^U\right], \left[\upsilon_{\alpha_i}^L, \upsilon_{\alpha_i}^U\right] \right\rangle, \langle \eta_{\alpha_i}, \upsilon_{\alpha_i} \rangle\right)$ $(i = 1, 2, 3)$ be any three CFFNs. Properties of the $\widetilde{CS}$ are:

1. $\mathscr{C}_{CFF}(\alpha_1, \alpha_2) = 1$ if and only if $\alpha_1 = \alpha_2$.
2. $\mathscr{C}_{CFF}(\alpha_1, \alpha_2) = \mathscr{C}_{CFF}(\alpha_2, \alpha_1)$.
3. $0 \leqslant \mathscr{C}_{CFF}(\alpha_2, \alpha_1) \leqslant 2$.

Proof. simple to demonstrate.

Definition 12. Let $\vartheta = \left(\left\langle \left[\eta_{\vartheta_i}^L, \eta_{\vartheta_i}^U\right], \left[\upsilon_{\vartheta_i}^L, \upsilon_{\vartheta_i}^U\right] \right\rangle, \langle \eta_{\vartheta_i}, \upsilon_{\vartheta_i} \rangle\right)$ and $\delta = \left(\left\langle \left[\eta_{\delta_i}^L, \eta_{\delta_i}^U\right], \left[\upsilon_{\delta_i}^L, \upsilon_{\delta_i}^U\right] \right\rangle, \langle \eta_{\delta_i}, \upsilon_{\delta_i} \rangle\right)$ be any two CFFNs, and take the weight vector ω_i. The weighted cosine similarity ($\widetilde{WCS}$) measure is defined as:

$$\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}\omega_i\left(\frac{A_i^L+A_i^U+B_i^L+B_i^U+C_i^L+C_i^U+A_i+B_i+C_i}{\sqrt[3]{a_i+b_i+c_i+d_i}\sqrt[3]{\widetilde{a}_i+\widetilde{b}_i+\widetilde{c}_i+\widetilde{d}_i}}\right) \quad (14)$$

where $A_i^L=\left(\eta^L_{\vartheta_i}\right)^3\left(\eta^L_{\delta_i}\right)^3$, $A_i^U=\left(\eta^U_{\vartheta_i}\right)^3\left(\eta^U_{\delta_i}\right)^3$, $B_i^L=\left(\upsilon^L_{\vartheta_i}\right)^3\left(\upsilon^L_{\delta_i}\right)^3$, $B_i^U=\left(\upsilon^U_{\vartheta_i}\right)^3\left(\upsilon^U_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^U=\left(\pi^U_{\vartheta_i}\right)^3\left(\pi^U_{\delta_i}\right)^3$, $A_i=\left(\eta_{\vartheta_i}\right)^3\left(\eta_{\delta_i}\right)^3$, $B_i=(\upsilon_{\vartheta_i})^3(\upsilon_{\delta_i})^3$, $C_i=(\pi_{\vartheta_i})^3(\pi_{\delta_i})^3$, $a_i=\left(\eta^L_{\vartheta_i}\right)^3+\left(\eta^U_{\vartheta_i}\right)^3$, $b_i=\left(\upsilon^L_{\vartheta_i}\right)^3+\left(\upsilon^U_{\vartheta_i}\right)^3$, $c_i=\left(\pi^L_{\vartheta_i}\right)^3+\left(\pi^U_{\vartheta_i}\right)^3$, $d_i=\left(\eta_{\vartheta_i}\right)^3+(\upsilon_{\vartheta_i})^3+(\pi_{\vartheta_i})^3$, $\widetilde{a}_i=\left(\eta^L_{\delta_i}\right)^3+\left(\eta^U_{\delta_i}\right)^3$, $\widetilde{b}_i=\left(\upsilon^L_{\delta_i}\right)^3+\left(\upsilon^U_{\delta_i}\right)^3$, $\widetilde{c}_i=\left(\pi^L_{\delta_i}\right)^3+\left(\pi^U_{\delta_i}\right)^3$, and $\widetilde{d}_i=\left(\eta_{\delta_i}\right)^3+(\upsilon_{\delta_i})^3+(\pi_{\delta_i})^3$. The weight vector must comply with the following conditions:

1. $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$.
2. $\sum_{i=1}^{n}\omega_i=1$.

Theorem 2. Choose any two CFFNs α and β. then, the $\widetilde{WCS}$ measure $\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)$ complies with the following criteria:

1. $\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)=1$ if and only if $\alpha=\beta$.
2. $\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)=\mathscr{C}^{\omega}_{CFF}(\beta,\alpha)$.
3. $0\leqslant\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)\leqslant2$.

where $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$, and $\sum_{i=1}^{n}\omega_i=1$.
Proof. simple to demonstrate.

3.2. Euclidean distance measure between CFFNs

Definition 13. Let $\vartheta=\left(\left\langle\left[\eta^L_{\vartheta_i},\eta^U_{\vartheta_i}\right],\left[\upsilon^L_{\vartheta_i},\upsilon^U_{\vartheta_i}\right]\right\rangle,\left\langle\eta_{\vartheta_i},\upsilon_{\vartheta_i}\right\rangle\right)$ and $\delta=\left(\left\langle\left[\eta^L_{\delta_i},\eta^U_{\delta_i}\right],\left[\upsilon^L_{\delta_i},\upsilon^U_{\delta_i}\right]\right\rangle,\left\langle\eta_{\delta_i},\upsilon_{\delta_i}\right\rangle\right)$ be any two CFFNs. The Euclidean distance between ϑ and δ can be defined as follow:

$$D_{CFF}(\vartheta,\delta)=\left(\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c}\left|\left(\eta^L_{\vartheta_i}\right)^3-\left(\eta^L_{\delta_i}\right)^3\right|^2+\left|\left(\eta^U_{\vartheta_i}\right)^3-\left(\eta^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\upsilon^L_{\vartheta_i}\right)^3-\left(\upsilon^L_{\delta_i}\right)^3\right|^2+\left|\left(\upsilon^U_{\vartheta_i}\right)^3-\left(\upsilon^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\eta_{\vartheta_i}\right)^3-\left(\eta_{\delta_i}\right)^3\right|^2+\left|(\upsilon_{\vartheta_i})^3-(\upsilon_{\delta_i})^3\right|^2+\\+\left|\left(\pi^L_{\vartheta_i}\right)^3-\left(\pi^L_{\delta_i}\right)^3\right|^2+\left|\left(\pi^U_{\vartheta_i}\right)^3-\left(\pi^U_{\delta_i}\right)^3\right|^2\\+\left|(\pi_{\vartheta_i})^3-(\pi_{\delta_i})^3\right|^2\end{array}\right)\right)^{\frac{1}{2}} \quad (15)$$

The weighted Euclidean distance between ϑ and δ is defined as:

$$D^{\omega}_{CFF}(\vartheta,\delta)=\left(\frac{1}{6n}\sum_{i=1}^{n}\omega_i\left(\begin{array}{c}\left|\left(\eta^L_{\vartheta_i}\right)^3-\left(\eta^L_{\delta_i}\right)^3\right|^2+\left|\left(\eta^U_{\vartheta_i}\right)^3-\left(\eta^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\upsilon^L_{\vartheta_i}\right)^3-\left(\upsilon^L_{\delta_i}\right)^3\right|^2+\left|\left(\upsilon^U_{\vartheta_i}\right)^3-\left(\upsilon^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\eta_{\vartheta_i}\right)^3-\left(\eta_{\delta_i}\right)^3\right|^2+\left|(\upsilon_{\vartheta_i})^3-(\upsilon_{\delta_i})^3\right|^2+\\+\left|\left(\pi^L_{\vartheta_i}\right)^3-\left(\pi^L_{\delta_i}\right)^3\right|^2+\left|\left(\pi^U_{\vartheta_i}\right)^3-\left(\pi^U_{\delta_i}\right)^3\right|^2\\+\left|(\pi_{\vartheta_i})^3-(\pi_{\delta_i})^3\right|^2\end{array}\right)\right)^{\frac{1}{2}} \quad (16)$$

where $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3. For any two CFFNs ϑ and δ, the weighted Euclidean distance measure $D^{\omega}_{CFF}(\vartheta,\delta)$ complies with the following criteria:

1. $D^{\omega}_{CFF}(\vartheta,\delta)=1$ if and only if $\vartheta=\delta$.
2. $D^{\omega}_{CFF}(\vartheta,\delta)=D^{\omega}_{CFF}(\delta,\vartheta)$.
3. $0\leqslant D^{\omega}_{CFF}(\vartheta,\delta)\leqslant2$.

where $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$, and $\sum_{i=1}^{n}\omega_i=1$.
Proof. Easy to prove.

3.3. New similarity measures

Definition 14. Let $\vartheta=\left(\left\langle\left[\eta^L_{\vartheta_i},\eta^U_{\vartheta_i}\right],\left[\upsilon^L_{\vartheta_i},\upsilon^U_{\vartheta_i}\right]\right\rangle,\left\langle\eta_{\vartheta_i},\upsilon_{\vartheta_i}\right\rangle\right)$ and $\delta=\left(\left\langle\left[\eta^L_{\delta_i},\eta^U_{\delta_i}\right],\left[\upsilon^L_{\delta_i},\upsilon^U_{\delta_i}\right]\right\rangle,\left\langle\eta_{\delta_i},\upsilon_{\delta_i}\right\rangle\right)$ be any two CFFNs. Then a new similarity measure $S_{CFF}(\vartheta,\delta)$ between ϑ *and* δ can be defined as follow:

$$S_{CFF}(\vartheta,\delta)=\frac{\mathscr{C}_{CFF}(\vartheta,\delta)+1-D_{CFF}(\vartheta,\delta)}{6} \quad (17)$$

where $\mathscr{C}_{CFF}(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{A_i^L+A_i^U+B_i^L+B_i^U+C_i^L+C_i^U+A_i+B_i+C_i}{\sqrt[3]{a_i+b_i+c_i+d_i}\sqrt[3]{\widetilde{a_i}+\widetilde{b_i}+\widetilde{c_i}+\widetilde{d_i}}}\right)$
such that $A_i^L=\left(\eta^L_{\vartheta_i}\right)^3\left(\eta^L_{\delta_i}\right)^3$, $A_i^U=\left(\eta^U_{\vartheta_i}\right)^3\left(\eta^U_{\delta_i}\right)^3$, $B_i^L=\left(\upsilon^L_{\vartheta_i}\right)^3\left(\upsilon^L_{\delta_i}\right)^3$, $B_i^U=\left(\upsilon^U_{\vartheta_i}\right)^3\left(\upsilon^U_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^U=\left(\pi^U_{\vartheta_i}\right)^3\left(\pi^U_{\delta_i}\right)^3$, $A_i=\left(\eta_{\vartheta_i}\right)^3\left(\eta_{\delta_i}\right)^3$, $B_i=(\upsilon_{\vartheta_i})^3(\upsilon_{\delta_i})^3$, $C_i=(\pi_{\vartheta_i})^3(\pi_{\delta_i})^3$, $a_i=\left(\eta^L_{\vartheta_i}\right)^3+\left(\eta^U_{\vartheta_i}\right)^3$, $b_i=\left(\upsilon^L_{\vartheta_i}\right)^3+\left(\upsilon^U_{\vartheta_i}\right)^3$, $c_i=\left(\pi^L_{\vartheta_i}\right)^3+\left(\pi^U_{\vartheta_i}\right)^3$, $d_i=\left(\eta_{\vartheta_i}\right)^3+(\upsilon_{\vartheta_i})^3+(\pi_{\vartheta_i})^3$, $\widetilde{a}_i=\left(\eta^L_{\delta_i}\right)^3+\left(\eta^U_{\delta_i}\right)^3$, $\widetilde{b}_i=\left(\upsilon^L_{\delta_i}\right)^3+\left(\upsilon^U_{\delta_i}\right)^3$, $\widetilde{c}_i=\left(\pi^L_{\delta_i}\right)^3+\left(\pi^U_{\delta_i}\right)^3$, and $\widetilde{d}_i=\left(\eta_{\delta_i}\right)^3+(\upsilon_{\delta_i})^3+(\pi_{\delta_i})^3$.
And

$$D_{CFF}(\vartheta,\delta)=\left(\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c}\left|\left(\eta^L_{\vartheta_i}\right)^3-\left(\eta^L_{\delta_i}\right)^3\right|^2+\left|\left(\eta^U_{\vartheta_i}\right)^3-\left(\eta^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\upsilon^L_{\vartheta_i}\right)^3-\left(\upsilon^L_{\delta_i}\right)^3\right|^2+\left|\left(\upsilon^U_{\vartheta_i}\right)^3-\left(\upsilon^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\eta_{\vartheta_i}\right)^3-\left(\eta_{\delta_i}\right)^3\right|^2+\left|(\upsilon_{\vartheta_i})^3-(\upsilon_{\delta_i})^3\right|^2+\\+\left|\left(\pi^L_{\vartheta_i}\right)^3-\left(\pi^L_{\delta_i}\right)^3\right|^2+\left|\left(\pi^U_{\vartheta_i}\right)^3-\left(\pi^U_{\delta_i}\right)^3\right|^2\\+\left|(\pi_{\vartheta_i})^3-(\pi_{\delta_i})^3\right|^2\end{array}\right)\right)^{\frac{1}{2}}$$

Definition 12. Let $\vartheta=\left(\left\langle\left[\eta^L_{\vartheta_i},\eta^U_{\vartheta_i}\right],\left[\upsilon^L_{\vartheta_i},\upsilon^U_{\vartheta_i}\right]\right\rangle,\left\langle\eta_{\vartheta_i},\upsilon_{\vartheta_i}\right\rangle\right)$ and $\delta=\left(\left\langle\left[\eta^L_{\delta_i},\eta^U_{\delta_i}\right],\left[\upsilon^L_{\delta_i},\upsilon^U_{\delta_i}\right]\right\rangle,\left\langle\eta_{\delta_i},\upsilon_{\delta_i}\right\rangle\right)$ be any two CFFNs. Then a new weighted similarity measure $S^{\omega}_{CFF}(\vartheta,\delta)$ between ϑ and δ can be defined as follow:

$$S^{\omega}_{CFF}(\vartheta,\delta)=\frac{\mathscr{C}^{\omega}_{CFF}(\vartheta,\delta)+1-D^{\omega}_{CFF}(\vartheta,\delta)}{6} \quad (18)$$

where $\mathscr{C}^{\omega}_{CFF}(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}\omega_i\left(\frac{A_i^L+A_i^U+B_i^L+B_i^U+C_i^L+C_i^U+A_i+B_i+C_i}{\sqrt[3]{a_i+b_i+c_i+d_i}\sqrt[3]{\widetilde{a_i}+\widetilde{b_i}+\widetilde{c_i}+\widetilde{d_i}}}\right)$ such that $A_i^L=\left(\eta^L_{\vartheta_i}\right)^3\left(\eta^L_{\delta_i}\right)^3$, $A_i^U=\left(\eta^U_{\vartheta_i}\right)^3\left(\eta^U_{\delta_i}\right)^3$, $B_i^L=\left(\upsilon^L_{\vartheta_i}\right)^3\left(\upsilon^L_{\delta_i}\right)^3$, $B_i^U=\left(\upsilon^U_{\vartheta_i}\right)^3\left(\upsilon^U_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^L=\left(\pi^L_{\vartheta_i}\right)^3\left(\pi^L_{\delta_i}\right)^3$, $C_i^U=\left(\pi^U_{\vartheta_i}\right)^3\left(\pi^U_{\delta_i}\right)^3$, $A_i=\left(\eta_{\vartheta_i}\right)^3\left(\eta_{\delta_i}\right)^3$, $B_i=(\upsilon_{\vartheta_i})^3(\upsilon_{\delta_i})^3$, $C_i=(\pi_{\vartheta_i})^3(\pi_{\delta_i})^3$, $a_i=\left(\eta^L_{\vartheta_i}\right)^3+\left(\eta^U_{\vartheta_i}\right)^3$, $b_i=\left(\upsilon^L_{\vartheta_i}\right)^3+\left(\upsilon^U_{\vartheta_i}\right)^3$, $c_i=\left(\pi^L_{\vartheta_i}\right)^3+\left(\pi^U_{\vartheta_i}\right)^3$, $d_i=\left(\eta_{\vartheta_i}\right)^3+(\upsilon_{\vartheta_i})^3+(\pi_{\vartheta_i})^3$, $\widetilde{a}_i=\left(\eta^L_{\delta_i}\right)^3+\left(\eta^U_{\delta_i}\right)^3$, $\widetilde{b}_i=\left(\upsilon^L_{\delta_i}\right)^3+\left(\upsilon^U_{\delta_i}\right)^3$, $\widetilde{c}_i=\left(\pi^L_{\delta_i}\right)^3+\left(\pi^U_{\delta_i}\right)^3$, and $\widetilde{d}_i=\left(\eta_{\delta_i}\right)^3+(\upsilon_{\delta_i})^3+(\pi_{\delta_i})^3$

$$D^{\omega}_{CFF}(\vartheta,\delta)=\left(\frac{1}{6n}\sum_{i=1}^{n}\omega_i\begin{pmatrix}\left|\left(\eta^L_{\vartheta_i}\right)^3-\left(\eta^L_{\delta_i}\right)^3\right|^2+\left|\left(\eta^U_{\vartheta_i}\right)^3-\left(\eta^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\upsilon^L_{\vartheta_i}\right)^3-\left(\upsilon^L_{\delta_i}\right)^3\right|^2+\left|\left(\upsilon^U_{\vartheta_i}\right)^3-\left(\upsilon^U_{\delta_i}\right)^3\right|^2\\+\left|\left(\eta_{\vartheta_i}\right)^3-\left(\eta_{\delta_i}\right)^3\right|^2+\left|(\upsilon_{\vartheta_i})^3-(\upsilon_{\delta_i})^3\right|^2+\\+\left|\left(\pi^L_{\vartheta_i}\right)^3-\left(\pi^L_{\delta_i}\right)^3\right|^2+\left|\left(\pi^U_{\vartheta_i}\right)^3-\left(\pi^U_{\delta_i}\right)^3\right|^2\\+\left|(\pi_{\vartheta_i})^3-(\pi_{\delta_i})^3\right|^2\end{pmatrix}\right)^{\frac{1}{2}}$$

The weight vector must comply with the following conditions:

1. $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$.
2. $\sum_{i=1}^{n}\omega_i=1$.

Theorem 4. For any two CFFNs ϑ and δ, the weighted Euclidean distance measure $D^{\omega}_{CFF}(\vartheta,\delta)$ complies with the following criteria:

1. $S^{\omega}_{CFF}(\vartheta,\delta)=1$ if and only if $\vartheta=\delta$.
2. $S^{\omega}_{CFF}(\vartheta,\delta)=S^{\omega}_{CFF}(\delta,\vartheta)$.
3. $0\leqslant S^{\omega}_{CFF}(\vartheta,\delta)\leqslant2$.

Proof. Easy to prove.

3.4. TOPSIS approach to MCDM with CFFSs

This section aims to construct a TOPSIS method for MCDM using the concepts of the CFFS.

Given that the alternatives $s=\{s_1,s_2,\cdots,s_m\}$ are being evaluated by experts based on the criteria $r=\{r_1,r_2,\cdots,r_n\}$, which are expressed in terms of Fuzzy CFFSs $A_{ij}=\left(\left\langle\left[\eta^L_{A_{ij}},\eta^U_{A_{ij}}\right],\left[\upsilon^L_{A_{ij}},\upsilon^U_{A_{ij}}\right]\right\rangle,\left\langle\eta_{A_{ij}},\upsilon_{A_{ij}}\right\rangle\right)$ such that $0\leqslant\eta^L_{A_{ij}}$, $\eta^U_{A_{ij}}$, $\upsilon^L_{A_{ij}}$, $\upsilon^U_{A_{ij}}$, $\eta_{A_{ij}}$, $\upsilon_{A_{ij}}\leqslant1$, $\left(\eta^U_{A_{ij}}\right)^3+\left(\upsilon^U_{A_{ij}}\right)^3\leqslant1$ and $\left(\eta_{A_{ij}}\right)^3+\left(\upsilon_{A_{ij}}\right)^3\leqslant1$.

Consider the weight vector ω of criteria, subject to the following constraints.

1. $0\leqslant\omega_i\leqslant1$ for all $i\in\mathbb{N}$.
2. $\sum_{i=1}^{n}\omega_i=1$.

Then the cubic Fermatean fuzzy decision matrix $Z=\left(A_{ij}\right)_{m\times n}=\left(\left\langle\left[\eta^L_{A_{ij}},\eta^U_{A_{ij}}\right],\left[\upsilon^L_{A_{ij}},\upsilon^U_{A_{ij}}\right]\right\rangle,\left\langle\eta_{A_{ij}},\upsilon_{A_{ij}}\right\rangle\right)_{m\times n}$ can be represented as:

$$Z=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1n}\\A_{21}&A_{22}&\cdots&A_{2n}\\\vdots&\vdots&\ddots&\vdots\\A_{m1}&A_{m2}&\cdots&A_{mn}\end{pmatrix}\tag{19}$$

where s_{ij} $(i=1,2,\cdots,m;j=1,2,\cdots,n)$ are CFFNs.

The proposed algorithm based on the suggested similarity measures is constructed as follows:

Step 1. In decision-making, criteria are used to evaluate and compare alternatives. There are two types of criteria: benefit criteria and cost criteria. Benefit criteria refer to the positive aspects of the alternatives being evaluated. They measure the advantages or benefits of each alternative and are often used to determine the best solution for a particular problem. For instance, when deciding on purchasing a car, benefit criteria may include factors such as comfort, fuel efficiency, and reliability. Cost criteria, on the other hand, refer to the negative aspects of the expenses associated with each alternative. These criteria measure the disadvantages or costs of each alternative and are important in making an informed decision as they can impact the overall feasibility of a solution. For example, in the same car purchase decision, cost criteria may include factors such as the purchase price, maintenance costs, and insurance costs.

Initially, the decision matrix $Z=\left(A_{ij}\right)_{m\times n}=\left(\left\langle\left[\eta^L_{A_{ij}},\eta^U_{A_{ij}}\right],\left[\upsilon^L_{A_{ij}},\upsilon^U_{A_{ij}}\right]\right\rangle,\left\langle\eta_{A_{ij}},\upsilon_{A_{ij}}\right\rangle\right)_{m\times n}$ will be normalized. The following negation operator will be utilized for normalization.

$$\widehat{Z}=\begin{cases}\left(\left\langle\left[\eta^L_{A_{ij}},\eta^U_{A_{ij}}\right],\left[\upsilon^L_{A_{ij}},\upsilon^U_{A_{ij}}\right]\right\rangle,\left\langle\eta_{A_{ij}},\upsilon_{A_{ij}}\right\rangle\right) forbenefit-typecriteria\\\left(\left\langle\left[\upsilon^L_{A_{ij}},\upsilon^U_{A_{ij}}\right],\left[\eta^L_{A_{ij}},\eta^U_{A_{ij}}\right]\right\rangle,\left\langle\upsilon_{A_{ij}},\eta_{A_{ij}}\right\rangle\right) forcost-typecriteria\end{cases}\tag{20}$$

The operator is defined as follows: If the criterion being evaluated is of the benefit type, no action is required. If the criterion is of the cost type, it will be transformed into a benefit-type criterion.

Step 2. The positive s^+ and negative s^- ideal solutions will be determined using the score and accuracy functions. These ideal solutions represent the best and worst possible outcomes, respectively, and will be used to evaluate the alternatives being considered. Where $s^+=\{s_1^+,s_2^+,\cdots,s_2^+\}$, $A^-=\{s_1^-,s_2^-,\cdots,s_2^-\}$. For $j=1,2,\cdots,n$ we have

$$s_j^+=\max\{Sc(s_{1j}),Sc(s_{2j}),\cdots,Sc(s_{nj})\}$$

and

$$s_j^-=\min\{Sc(s_{1j}),Sc(s_{2j}),\cdots,Sc(s_{nj})\}$$

If all of the score values are equal, the accuracy values must be used for comparison purposes. This is because accuracy values provide an additional basis for comparison when score values are identical.

Step 3. To calculate the separation for each alternative between the derived positive ideal solution s^+ and negative ideal solution s^- using the proposed distance measure D^{ω}_{CFF}. The separation is computed as follows:

$$D^{\omega}_{CFF}(s_i, s^+) = \sum_{j=1}^{n} \omega_i D^{\omega}_{CFF}(s_{ij}, s^+) \quad (21)$$

$$D^{\omega}_{CFF}(s_i, A^-) = \sum_{j=1}^{n} \omega_i D^{\omega}_{CFF}(s_{ij}, s^-) \quad (22)$$

Based on these separation measures, the closeness index σ_i associated with alternative s_i will be calculated as follows:

$$\sigma_i = \frac{D^{\omega}_{CFF}(s_i, s^+)}{D^{\omega}_{CFF}(s_i, s^+) + D^{\omega}_{CFF}(s_i, s^-)} \quad (23)$$

Step 4. To determine the relative measures of the alternatives, we will evaluate their σ_i values. The σ_i value is a measure of the suitability of each alternative for our purpose. The smaller the σ_i value, the better the alternative is considered to be. With this in mind, we will select the alternative s_i with the smallest δ_i value as the optimal choice, as it is deemed to offer the most satisfactory outcome based on the criteria established by the σ_i value.

3.5. Application

Numerous diseases exist, and each disease presents its own distinct set of symptoms. The process of medical diagnosis relies heavily on analyzing the symptoms exhibited by a patient to identify the specific disease they are experiencing. The patient's symptoms can be viewed collectively as a symptom set, and the diseases that match these symptoms form a diagnostic set. By examining the relationship between a patient's symptom set and possible diagnostic sets, medical professionals can effectively diagnose and treat a wide range of illnesses. However, it is important to note that the complexity of certain diseases may require additional tests or evaluations beyond just analyzing the patient's symptoms.

Let R be the set of symptom $R = \{r_1(\text{Temperature}), r_2(\text{Headache}), r_3(\text{Stompach pain}), r_4(\text{Cough}), r_5(\text{Chestpain})\}$ and the diagnostic set be $S = \{s_1(\text{Viral fever}), s_2(\text{Malaria}), s_3(\text{Typhoid}), s_4(\text{Stompach problem}), s_5(\text{Chest problem})\}$. The symptoms exhibited by a patient can be expressed using the CFFSs in the following manner.

$\mathscr{P}$(Patiment)

$$= \left\{ \begin{array}{c} (r_1, \langle[0.5,0.6],[0.2,0.4]\rangle, \langle 0.6,0.3\rangle), (r_2, \langle[0.4,0.5],[0.3,0.5]\rangle, \langle 0.2,0.6\rangle), \\ (r_3, \langle[0.3,0.4],[0.6,0.7]\rangle, \langle 0.5,0.6\rangle), (r_1, \langle[0.7,0.8],[0.5,0.6]\rangle, \langle 0.4,0.5\rangle), \\ (r_4, \langle[0.4,0.5],[0.6,0.7]\rangle, \langle 0.6,0.4\rangle) \end{array} \right\}.$$

The CFFSs can be used to describe the symptoms of each disease s_i (where i ranges from 1 to 5) in the following manner.

s_1(Viral fever)

$$= \left\{ \begin{array}{c} (s_1, \langle[0.3,0.4],[0.6,0.7]\rangle, \langle 0.2,0.5\rangle), (s_2, \langle[0.5,0.6],[0.4,0.6]\rangle, \langle 0.6,0.5\rangle), \\ (s_3, \langle[0.5,0.6],[0.3,0.5]\rangle, \langle 0.4,0.3\rangle), (s_4, \langle[0.4,0.5],[0.6,0.7]\rangle, \langle 0.3,0.4\rangle), \\ (s_5, \langle[0.5,0.6],[0.4,0.5]\rangle, \langle 0.8,0.1\rangle) \end{array} \right\},$$

s_2(Malaria)

$$= \left\{ \begin{array}{c} (s_1, \langle[0.5,0.6],[0.3,0.4]\rangle, \langle 0.6,0.1\rangle), (s_2, \langle[0.3,0.4],[0.5,0.6]\rangle, \langle 0.5,0.4\rangle), \\ (s_3, \langle[0.6,0.7],[0.5,0.6]\rangle, \langle 0.4,0.2\rangle), (s_4, \langle[0.5,0.6],[0.6,0.7]\rangle, \langle 0.7,0.3\rangle), \\ (s_5, \langle[0.5,0.6],[0.2,0.3]\rangle, \langle 0.8,0.2\rangle) \end{array} \right\}$$

,

s_3(Typhoid)

$$= \left\{ \begin{array}{c} (s_1, \langle[0.2,0.3],[0.6,0.7]\rangle, \langle 0.4,0.6\rangle), (s_2, \langle[0.3,0.4],[0.5,0.6]\rangle, \langle 0.5,0.4\rangle), \\ (s_3, \langle[0.6,0.7],[0.4,0.5]\rangle, \langle 0.3,0.2\rangle), (s_4, \langle[0.5,0.6],[0.3,0.4]\rangle, \langle 0.6,0.3\rangle), \\ (s_5, \langle[0.4,0.6],[0.4,0.5]\rangle, \langle 0.7,0.4\rangle) \end{array} \right\}$$

,

s_4(Stompach problem)

$$= \left\{ \begin{array}{c} (s_1, \langle[0.5,0.6],[0.6,0.7]\rangle, \langle 0.4,0.6\rangle), (s_2, \langle[0.5,0.6],[0.2,0.3]\rangle, \langle 0.5,0.3\rangle), \\ (s_3, \langle[0.4,0.5],[0.5,0.6]\rangle, \langle 0.2,0.7\rangle), (s_4, \langle[0.6,0.7],[0.1,0.3]\rangle, \langle 0.5,0.7\rangle), \\ (s_5, \langle[0.4,0.5],[0.6,0.7]\rangle, \langle 0.3,0.8\rangle) \end{array} \right\},$$

s_5(Chest problem)

$$= \left\{ \begin{array}{c} (s_1, \langle[0.3,0.6],[0.4,0.5]\rangle, \langle 0.5,0.7\rangle), (s_2, \langle[0.3,0.4],[0.6,0.7]\rangle, \langle 0.7,0.6\rangle), \\ (s_3, \langle[0.5,0.7],[0.3,0.5]\rangle, \langle 0.6,0.8\rangle), (s_4, \langle[0.2,0.5],[0.4,0.5]\rangle, \langle 0.3,0.2\rangle), \\ (s_5, \langle[0.6,0.7],[0.4,0.5]\rangle, \langle 0.5,0.4\rangle) \end{array} \right\}$$

The entropy measure is utilized in determining the weights of criteria through the following method:

$$\mathscr{E}_j = \frac{1}{(\sqrt{2}-1)\mathfrak{m}} \sum_{i=1}^{\mathfrak{m}} \left(1 + \eta_{ij}^2 - \upsilon_{ij}^2\right) \quad (24)$$

where $\frac{1}{(\sqrt{2}-1)\mathfrak{m}}$ is a constant for ensuring that $0 \leqslant \mathscr{E}_j \leqslant 1$ $(j = 1, 2, \cdots m)$. Determine the attribute weights ω_j using Equation (25).

$$\omega_j = \frac{1 - \mathscr{E}_j}{\mathfrak{n} - \sum_{j=1}^{\mathfrak{n}} \mathscr{E}_j} \quad (25)$$

The weighted vector for criteria H_i $(i = 1, 2, 3, 4, 5)$ is $(0.2407, 0.2130, 0.1543, 0.1859, 0.2061)$ obtained through the utilization of Equations (24) and (25).

The score values of each rating value can be determined through the use of equation (5), which can then be used to calculate the negative s^- and positive ideal s^+ solutions.

$$s^+ = \left\{ \begin{array}{c} (\langle[0.5,0.6],[0.4,0.5]\rangle, \langle 0.8,0.1\rangle), (\langle[0.5,0.6],[0.2,0.3]\rangle, \langle 0.8,0.2\rangle), \\ (\langle[0.6,0.7],[0.4,0.5]\rangle, \langle 0.3,0.2\rangle), (\langle[0.6,0.7],[0.1,0.3]\rangle, \langle 0.5,0.7\rangle), \\ (\langle[0.6,0.7],[0.4,0.5]\rangle, \langle 0.5,0.4\rangle) \end{array} \right\}$$

$$s^- = \left\{ \begin{array}{c} (\langle[0.3,0.4],[0.6,0.7]\rangle, \langle 0.2,0.5\rangle), (\langle[0.3,0.4],[0.5,0.6]\rangle, \langle 0.5,0.4\rangle), \\ (\langle[0.2,0.3],[0.6,0.7]\rangle, \langle 0.4,0.6\rangle), (\langle[0.4,0.5],[0.6,0.7]\rangle, \langle 0.3,0.8\rangle), \\ (\langle[0.3,0.4],[0.6,0.7]\rangle, \langle 0.7,0.6\rangle) \end{array} \right\}$$

We utilize Equation (23) to calculate the difference between positive ideal and negative ideal solutions for each alternative. This method involves the application of the closeness index σ_i to all values of s_i, to measure the separation of each alternative between the positive ideal and negative ideal solutions. The results and ranking orders are summarized in Table 3.

The results in Table 3 indicate that s_2 has the highest ranking among s_1, s_3, s_4, and s_5 based on the weighted similarity measures. The comparison between the new and existing weighted similarity measures shows that the similarity measure between $\mathscr{P}$ and s_2 is the lowest, leading to the diagnosis of malaria in the patient.

Table 3 Closeness index and Ranking of alternatives.

Closeness index $\boldsymbol{\sigma_i}$	Results	Ranking
$\boldsymbol{\sigma_1}$	0.4511	2
$\boldsymbol{\sigma_2}$	0.4152	1
$\boldsymbol{\sigma_3}$	0.4783	3
$\boldsymbol{\sigma_4}$	0.5217	5
$\boldsymbol{\sigma_5}$	0.4669	4

3.6. Comparative study

To facilitate a comparison between the proposed similarity measures and the existing ones, let us begin by briefly reviewing all the similarity measures that were evaluated by Jeevaraj [40], and Mishra [41].

3.6.1. Comparison with similarity measure by Jeevaraj [40]

Assuming Q_1 and Q_2 are two interval-valued intuitionistic fuzzy numbers, their similarity measure can be expressed as:

$$S(Q_1, Q_2) = 1 - D(Q_1, Q_2) \tag{26}$$

$$D(Q_1, Q_2) = \frac{3}{4}|Sc(Q_1) - Sc(Q_1)|$$

The use of similarity measures for interval-valued intuitionistic fuzzy sets stated in Equation (24), is not suitable for handling the rating values provided by the decision-makers listed in Table 2. This is because the condition $\eta^U + \upsilon^U \preccurlyeq 1$, which is required for the applicability of the similarity measures, is not satisfied. This limitation may hinder the accuracy of the results obtained from decision-making processes that involve the use of interval-valued intuitionistic fuzzy sets. Therefore, alternative methods or adjustments to the similarity measures may need to be considered to effectively handle such rating values. Further research may be necessary to explore and develop these alternatives.

3.6.2. Comparison with similarity measures by Mishra et al. [41]

Let $\vartheta = \left(\left\langle\left[\eta^L_{\vartheta_i}, \eta^U_{\vartheta_i}\right], \left[\upsilon^L_{\vartheta_i}, \upsilon^U_{\vartheta_i}\right]\right\rangle\right)$ and $\delta = \left(\left\langle\left[\eta^L_{\delta_i}, \eta^U_{\delta_i}\right], \left[\upsilon^L_{\delta_i}, \upsilon^U_{\delta_i}\right]\right\rangle\right)$ be any two IVPFNs. Then similarity measure $S_{IVPF}(\vartheta, \delta)$ between ϑ and δ can be defined as follow:

$$S_{IVPF}(\vartheta,\delta) = 1 - \frac{1 - exp\left(\frac{1}{4n}\sum_{i=1}^{n}\left(\begin{array}{c}\left|\left(\eta^L_{\vartheta_i}\right)^2 - \left(\eta^L_{\delta_i}\right)^2\right| + \left|\left(\eta^U_{\vartheta_i}\right)^2 - \left(\eta^U_{\delta_i}\right)^3\right| + \left|\left(\upsilon^L_{\vartheta_i}\right)^2 - \left(\upsilon^L_{\delta_i}\right)^3\right| + \\ \left|\left(\upsilon^U_{\vartheta_i}\right)^2 - \left(\upsilon^U_{\delta_i}\right)^2\right|^2 + \left|\left(\pi^L_{\vartheta_i}\right)^3 - \left(\pi^L_{\delta_i}\right)^3\right| + \left|\left(\pi^U_{\vartheta_i}\right)^3 - \left(\pi^U_{\delta_i}\right)^3\right|\end{array}\right)\right)}{1 - \exp(-1)} \tag{27}$$

To compare the proposed approach to the current similarity measure defined in Equation (25), we assume that the Fermatean fuzzy judgments of Cubic Fermatean Fuzzy are equal to zero i.e., $(\langle[\eta^L_\delta, \eta^U_\delta], [\upsilon^L_\delta, \upsilon^U_\delta]\rangle, \langle 0, 0\rangle)$.

Table 4 Closeness index of alternatives.

closeness index $\boldsymbol{\sigma_i}$	Results	Ranking
$\boldsymbol{\sigma}_1$	0.9016	2
$\boldsymbol{\sigma}_2$	0.8886	1
$\boldsymbol{\sigma}_3$	0.9221	3
$\boldsymbol{\sigma}_3$	0.9443	5
$\boldsymbol{\sigma}_3$	0.9316	4

By using Equation (25), the relative closeness index σ_i of alternatives A_i $(i = 1, 2, 3)$ is summarized in Table 4.

Table 4 shows that the proposed approach aligns with the best alternative, which confirms the approach's stability compared to state-of-the-art methods. The proposed decision-making method, which operates in a cubic Fermatean fuzzy set environment, offers more comprehensive evaluation information on alternatives than existing approaches that rely on interval-valued Pythagorean sets (IVPFS) or interval-valued Fermatean fuzzy sets (IVFFS). The proposed cosine similarity measures consider both interval-valued Fermatean fuzzy and Fermatean fuzzy sets simultaneously, while the existing similarity measures include either IVPFS or IVFFS information only. Consequently, the approaches based on IVPFSs or IVFFS may not capture all the relevant information about the alternatives, which could potentially affect the decision outcomes.

3.7. Advantages of the proposed approach

1. The key feature of CFFS systems is that the sum of cubes of the membership and non-membership values of any given object can be less than or equal to 1. This characteristic enables CFFSs to cover a larger number of elements compared to that CPFS and CIFS systems. Hence, the CFFS model represents a valuable and practical expansion of CIFSs and CPFSs, providing experts with greater autonomy in expressing their opinions regarding the degree of membership.
2. The proposed decision-making method under the CFFS environment is a more comprehensive approach compared to the existing methods that rely on either interval-valued FFSs (IVFFSs) or FFSs. This is because it takes into consideration both IVFFSs and FFSs to provide a more comprehensive evaluation of the alternatives. On the other hand, the existing methods that only consider either IVFFSs or FFSs may lack important information, such as interval-valued Fermatean fuzzy numbers or Fermatean fuzzy numbers, about the alternatives, which could potentially impact the accuracy of the decision results.
3. The process of selecting the best alternative from a set of alternatives in a MCDM problem is hindered when the uncertain data is forced to conform to the limited forms of CIFNs and CPFNs. This can result in the degradation of the data. To overcome this limitation, a more generalized model is necessary to provide reliable solutions in these critical situations. CFFSs provide more accurate and precise

results in addressing practical MCDM problems that involve cubic Fermatean fuzzy information, as they serve as a more effective extension of CIFSs and CPFSs.

4. The study that is being discussed does not limit its examination to only cosine similarity measures. It has also taken into account the use of Euclidean distance measures. By considering both measures, the study provides a comprehensive examination of the MCDM problem, incorporating both a geometric and an algebraic perspective. This allows for a more in-depth and nuanced understanding of the problem being investigated.

4. Conclusion

This research endeavours to address a Multi-Criteria Decision Making (MCDM) issue. In this MCDM problem, the authors consider two measures: The Cosine Similarity measure and the Cosine Distance between CFFSs. The CFFS values are used to define the cosine similarity measure and Euclidean Distance Measure. Furthermore, the fundamental characteristics of these measures are thoroughly analyzed and examined. Consequently, the authors have established novel Similarity Measures between the CFFSs based on the proposed Cosine Similarity Measure and Euclidean Distance Measure. These SMs not only fulfil the criteria for a similarity measure, but they also address the associated decision-making problems from both a geometric and algebraic perspective. The efficacy, impact, and adaptability of this method have been demonstrated through its application in a medical case study. In the future, we aim to broaden the range of applications of our proposed approach to encompass various fields [42 –44].

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Data Availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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