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ORIGINAL ARTICLE

A method to construct all the paving matroids over a finite set

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Abstract

We give a characterization of paving matroids through their sets of hyperplanes and an algorithm to construct all of them. We also give a simple proof of Rota's basis conjecture for the case of sparse-paving matroids and for the case of paving matroids of rank r on a set of cardinality $n \leq 2r$, and a counterexample to Oxley's characterization of paving matroids.

Keywords Simple matroid · Paving matroid · Sparse-paving matroid · Lattice · Hyperplanes of a matroid · Circuits of a matroid · Rota's basis conjecture

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1 Introduction

In 1959, Hartmanis [12] defined paving matroids through the concept of d-partitions in number theory. For a modern treatment of paving matroids, the reader may consult Welsh [25], Oxley [21] and Jerrum [15]. Paving matroids play an important role in computer science via greedy algorithms and the matroid oracles, see for example Heunen and Patta [13]. In this work, we give a characterization of paving matroids which leads to a concrete construction of their hyperplanes and to an algorithm to find them. We also provide a counterexample to a characterization of paving matroids given by Oxley [21] (1.3.10). Finally, we give a simple proof of Rota's basis conjecture for the case of sparse-paving matroids and for the case of paving matroids of rank r on a set of cardinality $n \leq 2r$.

In this paper, all matroids are assumed to be simple, so they do not have circuits of cardinality 1. We start with a characterization of the hyperplanes of a paving matroid of rank r.

Lemma *2 A simple matroid M of rank $r \geq 2$ is a paving matroid if and only if the intersection of any two different hyperplanes of M of cardinality at least r has cardinality at most $r-2$.*

Since the hyperplanes of cardinality at least r of a sparse-paving matroid are exactly the circuits of rank $r-1$, which have cardinality r, we get the following,

Corollary *[18] A simple matroid M of rank $r \geq 3$ is sparse-paving if and only if the intersection of any two different circuits of cardinality r of M has cardinality at most $r-2$.*

The next result gives a concrete construction of the hyperplanes of a paving matroid.

Theorem *1 Let S be a set of cardinality n and $2 \leq r \leq n$. Let $\mathcal{H}'$ be a nonempty family of subsets of S with cardinalities between r and $n-1$, such that the intersections of different elements in $\mathcal{H}'$ have cardinalities at most $r-2$. Let $\mathcal{C}_r$ be the family of subsets of S of cardinality r which are contained in an element of $\mathcal{H}'$ and let $\mathcal{H}$ be the union of $\mathcal{H}'$ and the family of subsets of S of cardinality $r-1$ which are not contained in elements of $\mathcal{C}_r$. Then, $\mathcal{H}$ is the set of hyperplanes of a paving matroid of rank r on S, and $\mathcal{C}_r$ is the set of circuits of cardinality r.*

Proposition *2 Let $\mathcal{P}av_{n,r}$ be the set of paving matroids of rank r on a set of cardinality n and let $Sp_{n,t}$ be the set of sparse-paving matroids of rank t on that set. Then, $\left|\mathcal{P}av_{n,r}\right| \leq \prod_{t=r}^{n-1} \left|Sp_{n,t}\right| \leq \left|Sp_{n,\frac{n}{2}}\right|^{n-r}$.*

This paper is organized as follows. In Sect. 3, we give a characterization of paving matroids by their sets of hyperplanes, a concrete construction of the hyperplanes of paving matroids and some consequences. In Sect. 4, we give an algorithm to construct the set of hyperplanes of a paving matroid of rank r on a set of cardinality n and prove a relation between the number of paving matroids and the number of sparse-paving matroids of rank $r \geq 3$ on a set of cardinality n. In Sect. 5 we give a new proof of Rota's basis conjecture.

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2 Definitions and known results

For later reference, we summarize some definitions and basic results.

A matroid $M = (S, \mathcal{I})$ consists of a finite set S and a collection $\mathcal{I}$ of subsets of S (called the independent sets of M) satisfying the following independence axioms:

($\mathcal{I}1$) The empty set $\emptyset \in \mathcal{I}$.

($\mathcal{I}2$) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

($\mathcal{I}3$) If $U, V \in \mathcal{I}$ with $|U| = |V| + 1$, there exists $x \in U \backslash V$ such that $V \cup \{x\} \in \mathcal{I}$.

A subset of S which does not belong to $\mathcal{I}$ is called a dependent set of M. A basis (respectively, a circuit) of M is a maximally independent (respectively, minimally dependent) set of M. Any matroid $M = (S, \mathcal{I})$ is determined by its set of bases, $\mathcal{B}$, namely, $\mathcal{I} = \{X \subseteq S : \exists B \in \mathcal{B} \text{ with } X \subseteq B\}$. The rank of a subset X of S, denoted by rk$X$, is the maximum cardinality among the independent subsets of X. The rank of the matroid M is the rank of S. A hyperplane of M is a maximal subset of rank rk$M - 1$. Any circuit of M has cardinality at most rk$M + 1$, and any hyperplane Y satisfies rk$M - 1 \leq |Y| \leq n - 1$.

If $M = (S, \mathcal{I})$ is a matroid, the dual of M is the matroid $M^* = (S, \mathcal{I}^*)$ whose bases are the complements in S of the bases of M.

A matroid M is paving if it has no circuits of cardinality less than rk$M$ and it is sparse-paving if M and its dual $M^*$are paving matroids. Well-known examples of sparse-paving matroids are the uniform matroids of rank r on a set S of cardinality n, denoted by $U_{n,r}$, whose bases are all the subsets of S of cardinality r. Any simple matroid of rank 1 must be uniform, so we will work with matroids of rank at least 2. For a set X and $m \geq 0$, $\binom{X}{m}$ denotes the family of subsets of X of cardinality m.

Definition 1 Let $M = (S, \mathcal{I})$ be a paving matroid of rank $r \geq 2$ on $S = \{1, 2, ..., n\}$. Let $\mathcal{B}$ be the set of bases of M and let $\mathcal{C}_r$ and $\mathcal{C}_{r+1}$ be the circuits of cardinality r and $r + 1$, respectively. Define

$$\mathcal{N}_1 = \left\{ X \in \binom{S}{r+1} \mid X = C \cup B \text{ for some } C \in \mathcal{C}_r \text{ and } B \in \mathcal{B} \right\} \tag{1}$$

$$\mathcal{N}_2 = \left\{ X \in \binom{S}{r+1} \mid \text{ for all } A \in \binom{X}{r}, A \in \mathcal{C}_r \right\}. \tag{2}$$

It is easy to show that the set C in the definition of $\mathcal{N}_1$ is unique.

Lemma 1 [18] *If $M=(S,\mathcal{I})$ is a paving matroid of rank r then $\binom{S}{r+1}=\mathcal{C}_{r+1}\cup\mathcal{N}_1\cup\mathcal{N}_2$. Moreover, M is a sparse-paving matroid if and only if $\mathcal{N}_2$ is the empty set.*

3 A description of the paving matroids through their sets of hyperplanes

Recall that a family $\mathcal{F}$ of two or more subsets of a set S is a d-partition if every set in $\mathcal{F}$ has cardinality at least d and every subset of cardinality d of the union of the elements of $\mathcal{F}$ is contained in exactly one set in $\mathcal{F}$.

In [25], Welsh characterized paving matroids as follows. A matroid $M=(S,\mathcal{I})$ of rank $d+1$ with $3\leq d+1<|S|$ is paving if its hyperplanes form a d-partition. See also [12]. We give another characterization.

Lemma 2 *A simple matroid M of rank $r\geq 2$ is a paving matroid if and only if the intersection of any two different hyperplanes of M of cardinality at least r has cardinality at most $r-2$.*

Proof Suppose that M is a paving matroid of rank r. Then the family of hyperplanes of M forms a $(r-1)$-partition. To prove assertion $(**)$, observe that if C_1 and C_2 are circuits of cardinality r with $|C_1\cap C_2|=r-1$, then $|C_1\cup C_2|=r+1$, and by Lemma 1 $C_1\cup C_2\in\mathcal{N}_2$. Then neither C_1 nor C_2 are hyperplanes, $\mathrm{rk}(C_1\cup C_2)=r-1$, and since the hyperplanes of M form a $(r-1)$-partition, there is a unique hyperplane H containing $C_1\cup C_2$. Therefore, if H_1 and H_2 are different hyperplanes of cardinality at least r then $\binom{H_1}{r-1}\cap\binom{H_2}{r-1}=\emptyset$ and $|H_1\cap H_2|\leq r-2$.

To prove that a matroid M of rank r satisfying property $(**)$ is paving, we will show that any subset of cardinality $r-1$ of S is an independent set. Assume that there is a dependent subset A of cardinality $r-1$. Then, $\mathrm{rk}A\leq r-2$, and since the rank of M is r, there are two different elements a, b in $S-A$ so that $A\cup\{a\}$ and $A\cup\{b\}$ have rank at most $r-1$. By $(**)$ neither $A\cup\{a\}$ nor $A\cup\{b\}$ are hyperplanes. So $A\cup\{a,b\}$ has also rank at most $r-1$, so there is another element x in $S-(A\cup\{a,b\})$ such that $A\cup\{a,x\}$ is a dependent set of rank at most $r-1$, and by $(**)$ neither $A\cup\{a,x\}$ nor $A\cup\{a,b\}$ are hyperplanes. In a finite number of steps, this procedure reaches S, a contradiction since $\mathrm{rk}S=r$. Therefore, every subset of cardinality $r-1$ is independent, and so M is paving. $\square$

The next result is the construction of the hyperplanes of a paving matroid.

Theorem 1 *Let S be a set of cardinality n and $2\leq r\leq n$. Let $\mathcal{H}'$ be a nonempty family of subsets of S with cardinalities between r and $n-1$, such that the intersections of different elements in $\mathcal{H}'$ have cardinalities at most $r-2$. Let $\mathcal{C}_r$ be the family of subsets of S of cardinality r which are contained in an element of $\mathcal{H}'$ and let $\mathcal{H}$ be the union of $\mathcal{H}'$ and the family of subsets of S of cardinality $r-1$ which are* *not contained in elements of $\mathcal{C}_r$. Then $\mathcal{H}$ is the set of hyperplanes of a paving matroid of rank r on S, and $\mathcal{C}_r$ is the set of circuits of cardinality r.*

Proof We first prove that $\mathcal{H}$ is an $(r-1)$-partition of S. Let $\mathcal{H}_{r-1}$ be the family of subsets of cardinality $r-1$ of S which are not contained in circuits of cardinality r. By construction of $\mathcal{H}_{r-1}$ and $\mathcal{H}'$, every subset of S of cardinality $r-2$ is contained in an element of $\mathcal{H}_{r-1}$ or $\mathcal{H}'$. Thus, S is the union of the elements in $\mathcal{H}$.

Now, we prove that for any subset A of cardinality $r-1$ of S, there is a unique element X in $\mathcal{H}$ containing A. By construction of $\mathcal{H}_{r-1}$, all its elements are hyperplanes of cardinality $r-1$. If A is not in $\mathcal{H}_{r-1}$ there exists C in $\mathcal{H}_r \cup \widetilde{\mathcal{C}}_r$ such that A is contained in C, where $\widetilde{\mathcal{C}}_r = \{C \in \binom{S}{r} \mid \exists X \in \mathcal{H}, |X| \geq r+1 \text{ and } C \subset X\}$ and $\mathcal{H}_r = \{C \in \mathcal{H} \mid |X| = r\}$. Now we consider two subcases:

If C is in $\mathcal{H}_r$ then by Lemma 2, for any X in $\mathcal{H} \backslash \{C\}$, $|X \cap C| \leq r-2$ and so A cannot be contained in X.

If C is in $\widetilde{\mathcal{C}}_r$, there exists X in $\mathcal{H}$ of cardinality at least $r+1$ containing C. Again by Lemma 2, for every Y in $\mathcal{H} \backslash \{X\}$, $|X \cap Y| \leq r-2$ and so A cannot be contained in Y.

Therefore, $\mathcal{H}$ is a $(r-1)$- partition of S and so $\mathcal{H}$ is the set of hyperplanes of a paving matroid on S. $\square$

As a consequence of the above results and since the hyperplanes of any matroid of rank r have cardinalities between $r-1$ and $n-1$, we get an injective function from the set of paving matroids of rank r on a set S of cardinality n into the direct product of the sets of sparse-paving matroids of rank k on S for $r \leq k \leq n-1$.

Proposition 2 *Let $\mathcal{P}av_{n,r}$ be the set of paving matroids of rank r on a set of cardinality n and let $Sp_{n,t}$ be the set of sparse-paving matroids of rank t on that set. Then*

$$\mathcal{P}av_{n,r} \overset{f}{\hookrightarrow} \prod_{t=r}^{n-1} Sp_{n,t} \text{ and therefore, } |\mathcal{P}av_{n,r}| \leq \prod_{t=r}^{n-1} |Sp_{n,t}| \leq |Sp_{n,\frac{n}{2}}|^{n-r}.$$

Proof Let M be a paving matroid of rank r on a set of cardinality n and let $\mathcal{H}_t$ be the family of hyperplanes of cardinality t of M. By Lemma 1 [18], $\mathcal{H}_t$ defines a sparse-paving matroid $M^{(t)}$ of rank t, whose set of circuits of cardinality t is $\mathcal{H}_t$. Define $f(M) = (M^{(t)})_t$. $\square$

Remark In [21](1.3.10), Oxley gives the following characterization of paving matroids: Let $\mathcal{D}$ be a collection of non-empty subsets of a set E. Then, $\mathcal{D}$ is the set of circuits of a paving matroid on E if and only if there is a positive integer k with $k \leq |E|$ and a subset $\mathcal{D}'$ of $\mathcal{D}$ such that

1. Every member of $\mathcal{D}'$ has k elements, and if two distinct members D_1 and D_2 of $\mathcal{D}'$ have $k-1$ common elements, then every subset of $D_1 \cup D_2$ with k elements is in $\mathcal{D}'$.
2. $\mathcal{D} - \mathcal{D}'$ consists of all subsets of E with $k+1$ elements that contain no member of $\mathcal{D}'$.

This characterization is not quite correct. If M is a paving matroid of rank k, $\mathcal{C}_k$ is the set of circuits of cardinality k and $\mathcal{C}_{k+1}$ the set of circuits of cardinality $k+1$ then $\mathcal{D} = \mathcal{C}_k \cup \mathcal{C}_{k+1}$ satisfies (1) and (2) with $\mathcal{D}' = \mathcal{C}_k$. But now, let $E = \{1,2,3,4\}$, $k = 3$ and let $\mathcal{D} = \mathcal{D}' = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ be all the subsets of cardinality 3 of E. Then, $\mathcal{D}$ satisfies (1) and (2), since every pair D_1 and D_2 of elements in $\mathcal{D}$ have intersection $k-1$ and $D_1 \cup D_2 = E$. But this construction cannot give a matroid of rank 3, since there is no basis.

4 An algorithm to construct the paving matroids

The algorithm below constructs a maximal set $\mathcal{H}_t$ of hyperplanes of cardinality $t \geq r$ of a paving matroid of rank r on a set S of cardinality n and the hyperplanes of cardinality $r-1$ are the sets $\mathcal{H}_{r-1} = \binom{S}{r-1} - \bigcup_{t=r}^{n-1} \mathcal{H}_t$.

Algorithm 1 PavingMatroids

```
H ← []
S ← [1, 2, ..., n]
P_S ← ℘(S)
flag ← 0
for m = r to n − 1 do
    for S_i in P_S such that |S_i| = m do
        flag ← 0
        for S_j in H do
            if |S_i ∩ S_j| > r − 2 then
                flag = 1
                break
            end if
        end for
        if flag == 0 then
            H ← H.adjoin(S_i)
        end if
    end for
end for
return H
```

5 Another proof of Rota's basis conjecture for sparse-paving matroids.

Rota's Basis Conjecture Let $B_1, \ldots, B_r$ be r disjoint bases of M. Let A be the $r \times r$-matrix with B_i as its ith-row, for $i = 1, \ldots, r$. Then, there exist a permutation in each row, such that all the columns of A are also bases of M.

In 1989, Rota published his basis conjecture [14] (conjecture 4), [24] (problem 1) which has an important role in matroid theory and is related to problems which can be solved if this conjecture is true. This motivated the weak and asymptotic versions of this conjecture, among others, see [1, 4, 6, 9] and [23]. In [10], Geelen and

Humphries proved it for paving matroids, and there are results for the asymptotic behaviour [8, 23] and for infinitely many values in real linear algebras [11].

We give a proof for sparse-paving matroids that is both simple and reveals the permutations needed to produce matrices whose columns form the desired bases. The proof also works for paving matroids of rank r on a set of cardinality $n \leq 2r$, since for permutations in the rows, the columns remain disjoint and the hyperplanes have cardinality at most $2r-1$, so there is at most one circuit of each hyperplane as a column of the matrix of the conjecture.

Proposition 3 *Rota's basis conjecture holds for sparse-paving matroids of rank at least 3.*

Proof Recall that in a sparse-paving matroid M all the circuits of rank $r-1$ have cardinality r, and a matroid M is sparse-paving if and only if for each pair of different circuits X and Y of rank $r-1$, $|X \cap Y| \leq r-2$ $(**)$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix}.$$

That is, $B_i = \{a_{i1}, a_{i2}, \ldots, a_{ir}\}$ is a basis of M, for $i = 1, \ldots, M$. We will consider four cases:

Case 1 If each column of A is already a basis of M, we are done.

Let ${}_{(ij,ik)}A$ be the matrix obtained from A by interchanging its (i,j) and (i,k) entries.

Case 2 If M has exactly one circuit of rank $r-1$ and this circuit is a column of A. We may assume that it is the first column of A. Since M has exactly one circuit of rank $r-1$ and we remove it from A, then the matrix ${}_{(11,12)}A$ has bases of M in all its columns, so we are done.

Case 3 Let M be a sparse-paving matroid with at least two different circuits of rank $r-1$. Let m be the number of columns of A which are circuits of M.

If A has at least $m \geq 2$ circuits of cardinality r, then we can assume that the circuits are in the first columns of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & \cdots & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{r-1,1} & a_{r-1,2} & \cdots & a_{r-1,m} & & a_{2r} \\ a_{r1} & a_{r2} & \cdots & a_{rm} & \cdots & a_{rr} \end{bmatrix}$$

Take the cyclic permutation $p: a_{r1} \longrightarrow a_{r2} \longrightarrow \cdots a_{r,m-1} \longrightarrow a_{rm} \longrightarrow a_{r1}$. Therefore, by $(**)$ and applying the permutation p, the resulting matrix

$$
{}_pA = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{r-1,1} & a_{r-1,2} & \cdots & a_{r-1,m} & a_{r-1,m+1} & & a_{2r} \\ a_{rm} & a_{r1} & \cdots & a_{r,m-1} & a_{r,m+1} & \cdots & a_{rr} \end{bmatrix}
$$

has bases of M in all its columns, and we are done.

Case 4 $M = (S, \mathcal{I})$ has at least two circuits of cardinality r and A has exactly one circuit. Using that $r \geq 3$ (ie., $|S| \geq 9$), we will prove that there is a permutation in some rows, such that the resulting matrix has at least 2 columns which are circuits of M and then we can apply Case 3 to get the result. We can assume that A has a circuit in its first column.

Proof of Case 4 By way of contradiction, assume that for all $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix}$ in $(S_r)^r$, where φ_i is a permutation of elements in the ith row of A, the resulting matrix, ${}_\varphi A$ has a unique column which is a circuit of M.

In particular, the assumption is true for all the transpositions $(i1, ij)$ with $j \neq 1$. Then, by $(**)$, the resulting matrix ${}_{(i1,ij)}A$ has a basis in the first column and therefore, the jth column is a circuit of M.

Let us construct ${}_{(21,22)}A$ and ${}_{(22,23)(31,33)}A$ from a matrix A whose unique circuit is in the first column. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ a_{41} & a_{42} & a_{43} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{r1} & a_{r2} & a_{r3} & \cdots \\ \Cap & \Cap & \Cap & \Cap \\ \mathcal{C}_r & \mathcal{B}_r & \mathcal{B}_r & \mathcal{B}_r \end{bmatrix}$ ${}_{(21,22)}A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{22} & a_{21} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ a_{41} & a_{42} & a_{43} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{r1} & a_{r2} & a_{r3} & \cdots \\ \Cap & \Cap & \Cap & \Cap \\ \mathcal{B}_r & \mathcal{C}_r & \mathcal{B}_r & \mathcal{B}_r \end{bmatrix}$

${}_{(31,33)}A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{33} & a_{32} & a_{31} & \cdots \\ a_{41} & a_{42} & a_{43} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{r1} & a_{r2} & a_{r3} & \cdots \\ \Cap & \Cap & \Cap & \Cap \\ \mathcal{B}_r & \mathcal{B}_r & \mathcal{C}_r & \mathcal{B}_r \end{bmatrix}$ ${}_{(22,23)(31,33)}A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{23} & a_{22} & \cdots \\ a_{33} & a_{32} & a_{31} & \cdots \\ a_{41} & a_{42} & a_{43} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{r1} & a_{r2} & a_{r3} & \cdots \\ \Cap & \Cap & \Cap & \Cap \\ \mathcal{B}_r & \mathcal{C}_r & \mathcal{B}_r & \mathcal{B}_r \end{bmatrix}$,

Then $\begin{bmatrix} a_{12} \\ a_{21} \\ a_{32} \\ a_{42} \\ \vdots \\ a_{r2} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{23} \\ a_{32} \\ a_{42} \\ \vdots \\ a_{r2} \end{bmatrix}$ are elements of $\mathcal{C}_r$ with intersection of cardinality $r - 1$, contradicting $(**)$.

Thus, there exists φ in $(S_r)^r$ such that ${}_{\varphi}A$ has more than one circuit of M and by Case 3, we are done. Therefore, Rota's basis conjecture is true for all sparse-paving matroids. □

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