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ARTICLE

The additive Perks distribution and its applications in reliability analysis

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ABSTRACT

In this paper, the Additive Perks Distribution (ADP) is presented; this distribution describes the behavior of the bathtub curve, which is one of the most deeply rooted concepts in reliability analysis. The proposed distribution is based on the sum of the positive and negative hazard rate functions of the Perks distribution. ADP's statistical properties include Measures of Central Tendency, moments, moment generating function, statistical order, residual lifetime, reversed residual lifetime, Rényi entropy, time-varying, and life-stress relationship modeling presented. For estimating the ADP parameters, the Maximum Likelihood Method is used. On the other hand, the APD is tested in three case studies and compared with other distributions that can describe the failure rates in a bathtub curve. The results from the case studies showed that the APD is a good choice for describing the failure times in the shape of a bathtub curve.

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1. Introduction

Reliability engineering has become a handy quality tool to determine the life behavior of any class of devices. One of the most widely used concepts in reliability is the description of device failures through the bathtub curve (see Figure 1). The concept of the bathtub curve lists that during the early stage of the device (infant mortality), failures can arise due to the manufacturing process or the quality of the components. The failures at this point behave in an exponentially decreasing way until stabilizing. When the device enters the stabilization stage, this is considered the operating use stage.

During this process, the failures are constant until the internal components reach their maximum life, so the probability of failure is low. Finally, the device enters the wear stage, where the failure is represented in an increasing exponential behavior; at this stage, the probability of failure is high, and the device can stop working correctly. Some distributions, such as the Weibull distribution (WD), Exponential Distribution (ED), and Lognormal Distribution (LD), among others, are the first choice for reliability practitioners to perform statistical analyzes due to the flexibility of these models. Nevertheless, these classical distributions show a constant or monotone failure rate, which breaks the assumption of bathtub curve behavior.

Researchers have proposed hybrid distributions that can represent the behavior of the bathtub shape curve with different applications towards reliability. For example, Aarset (1987) proposed the

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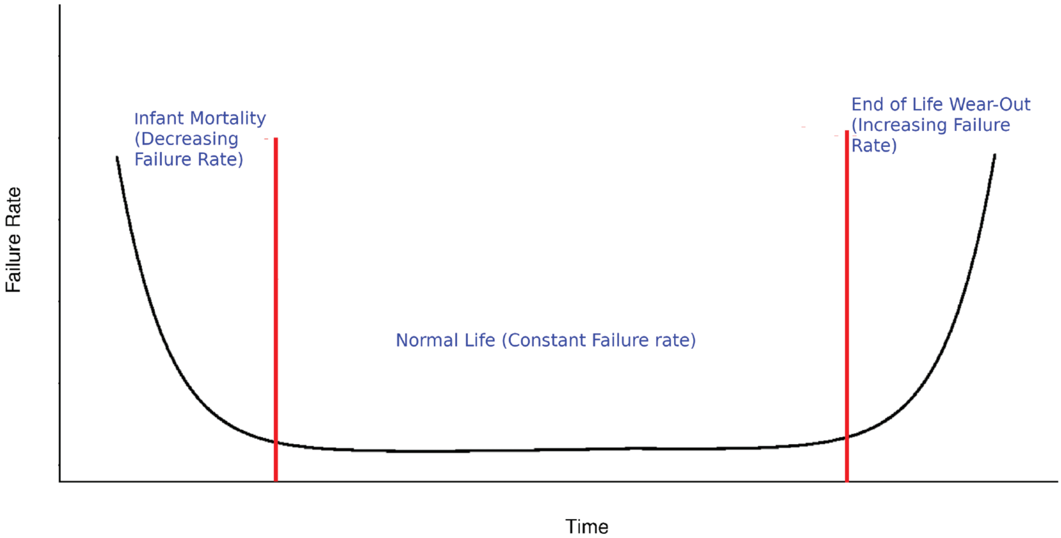


Figure 1. Reliability bathtub failure rate representation.

methodology to identify if the data have the property of a failure rate in an increasing, decreasing, or bathtub curve through the Total Time on Test (TTT). Mann et al. (1974) proposed a series of mixtures to the WD to improve the experimentation data results. Hjorth (1980) proposed a distribution with three parameters, with which it is possible to obtain Increasing, Decreasing, and Bathtub failure rate properties. Bebbington et al. (2007) introduced an extension of the WD by adding two parameters that allow data to be represented in the shape of a bathtub curve. Carrasco et al. (2008) analyzed a variation of four parameters from the WD. The proposed distribution has better flexibility to analyze the bathtub-shaped failure times than exponentiated and modified Weibull submodels functions. Other modifications that achieve a bathtub curve behavior are presented and analyzed by Pham and Lai (2007), Murthy et al. (2004), C. Lai (2013), Dey and Elshahhat (2022), Al Mutairi et al. (2021), Sharma et al. (2022), Benkhelifa (2022) and Krishna et al. (2022).

Similarly, some authors propose a mixture of two distributions from which a failure rate can be obtained in the form of a bathtub. Lai et al. (2016) introduced a new form of parameterization, which involves the representation of the hazard function as a generalized beta function ratio, with which it is possible to obtain bathtub curve representations. Lee et al. (2007) proposed the Beta-Weibull (BWD) distribution of four parameters, where the properties of the Beta and Weibull Distribution are combined for modeling a non-monotonic failure rate in reliability analysis. Nadarajah et al. (2013) studied the Exponentiated Weibull Distribution (EWD), which is a generalization of the two-parameter WD to accommodate non-monotone hazard rates. Mahdavi and Kundu (2017) introduced the Alpha Power Transformation (APT); this transformation proposes to add a new parameter to the family of exponential distributions. The proposed new parameter incorporates skewness, which allows the distributions to have better flexibility to represent non-monotonic behaviors. A practical case of the application of the APT with the WD where the reliability of electronic devices (ELD) is determined is reported by Méndez-González et al. (2022)

Other notable methodologies propose the sum of the risk functions; this sum can be combined with the same distribution or two different distributions. Following the additive methodology, Xie and Lai (1996) presented the addition of two risk functions of the WD, calling it the Additive Weibull (AWD) function. Xie et al. (2002) modified the WD with which they were able, with only two shape parameters and one scale parameter, to obtain a non-monotonic representation that resembles a bathtub curve, thus presenting the Modified Weibull Extension (MWE). The MWE is asymptotically related to the WD and the Exponential Distribution, giving it the flexible properties of these two distributions. Almalki and Yuan (2013) modified the AW proposed by Xie et al. by considering a serial system with one component called the New Modified Weibull (NMW). Zaindin and Sarhan (2009) proposed the Sarhan and Zaindin modified Weibull (SZMW), which has better flexibility to model the failure rate; the SZMW starts from being a special case of the WD. Other additive distributions can be seen in Abd EL-Baset and Ghazal (2020), He et al. (2016), Thach (2022) and Thanh Thach and Briš (2021).

The distributions presented in the literature can model the failure rate in a non-monotonic way. However, the observed shape tends to have a 'J' or a 'V' shape. This type of shape can be a problem in reliability analysis since it does not have a shape close to Figure 1. That is because the life of normal use is not adequately represented (in many cases is a very short portion of the hazard representation by the distribution). Therefore, the estimation and representation of the life of the device may not be the closest to reality.

The aforementioned establishes a need to explore new distributions that can describe the behavior of the devices closer to the real usage rate. The Perks distribution (PD) introduced by Perks (1932) has been widely used in the actuary branch to determine the mortality rate of people in insurance collection. In recent years, Singh and Choudhary (2017), Zeng et al. (2016), and Oguntunde et al. (2018) introduced methodologies that use PD in the analysis of the reliability of devices, managing to obtain shapes close to the bathtub curve. So, in this article, the Additive Perks Distribution (APD) is proposed as an alternative for the reliability analysis of devices with non-monotonic failure behavior. The APD is formulated from the sum of the positive and negative risk functions. The APD is more flexible with the experimental data that possess a non-monotonic failure than some distributions presented in the literature. The Mathematical flexibility shows that it can be a good option for reliability analysis.

The mentioned properties of the APD help to describe the lifetime behavior of some common devices in reliability analysis, such as electrical, electronic, and mechanical. These devices have life behaviors similar to those of living beings, where the probability of death decreases as the systems mature at early ages. However, at advanced ages, they may tend to a probability of death that may increase suddenly or remain constant under natural conditions. This type of behavior represents a problem for distributions that cannot change over time. For this reason, using a mortality model such as the one presented by Perks can be very useful for product, design, and maintenance engineers to determine the real behavior of the device under analysis, maintenance times, performance, and quality. On the other hand, the APD proposal uses the same mortality equation without combining it with any other known probability distribution in the study of product survival. The above is attractive for a clearer understanding of the different behaviors of the lifetimes data may have under other methodologies that propose properties similar to the APD.

In order to test the APD, three case studies were established where the APD is compared against other distributions of different construction methodologies that represent bathtub curve failures. To estimate the parameters of each of the distributions, the Maximum Likelihood Method (MLE) was used. The Akaike information criterion (AIC), the Bayesian information criterion (BIC), Kolmogorov Smirnov test (K-S), Anderson-Darling (AD), Cramér-von Mises (CVM), and P-value were used to compare the distributions under analysis.

Finally, this paper is organized as follows: Section 2 presents the APD model construction. Section 3 presents the Measures of Central Tendency of APD. Section 4 presents the order statistics. Section 5 presents the moments and moments generating function (MGF). Section 6 presents the residual and reversed residual lifetime function. Section 7 presents the Rényi Entropy. Section 8 presents an approach for time-varying modeling of APD and an Accelerated Life Testing (ALT) modeling centered on ELD. Section 9 presents the likelihood function to calculate the parameters proposed in section 2. Section 10 presents the case studies of the paper.

11 presents the conclusions of the manuscript and a future work proposal of the proposed statistical distribution.

2. APD model construction

Let $h_{pp}(x) = \frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}}$, the positive hazard rate function of PD. On the other hand let $h_{pn}(x) = \frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta x}}$, the negative hazard rate function of PD. Therefore, the risk function of the APD is defined by:

$$\begin{aligned} h(x) &= h_{pp}(\alpha;\lambda) + h_{pn}(\beta;\theta). \\ &= \left(\frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}} + \frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta x}}\right), \end{aligned} \tag{1}$$

where, α, β are the shape parameters and λ, θ are the scale parameters of the APD with $\alpha, \beta, \lambda, \theta > 0$.

The Probability Density Function (PDF) of APD from Eq.1 can be defined as:

$$\begin{aligned} f(x) &= h(x) \cdot exp\left(-\int_0^x h(v)dv\right), \\ &= \left(\frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}} + \frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta x}}\right) \cdot \frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})}. \end{aligned} \tag{2}$$

The Reliability Function $R(x)$ of the APD can be derived from the Eq.1 and Eq.2 and can be written as:

$$R(x) = \frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})}. \tag{3}$$

The Cumulative Density Function (CDF) can be written as:

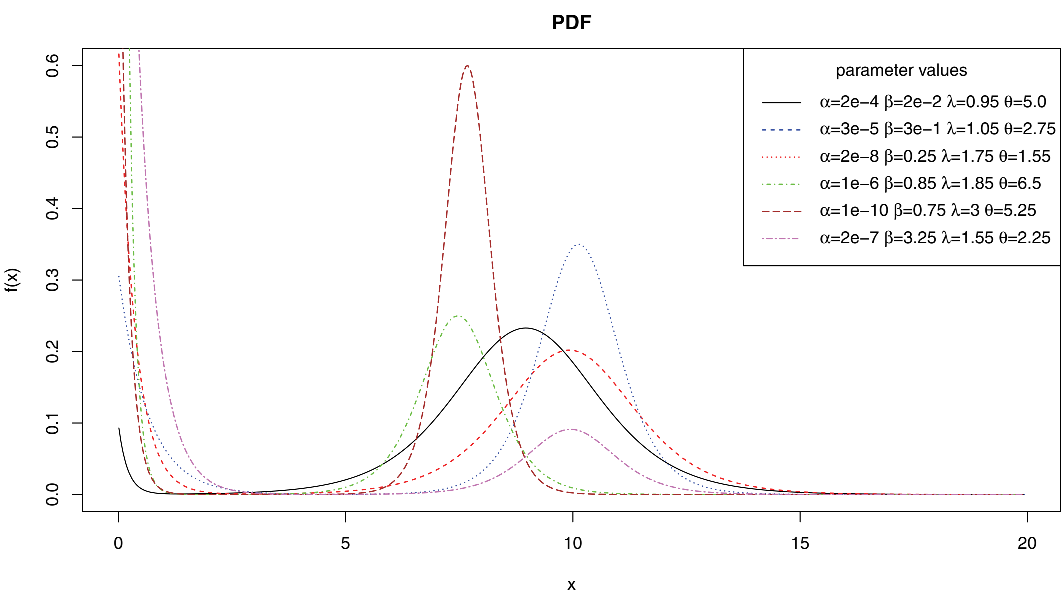


Figure 2. PDF for some parameter values derived of APD.

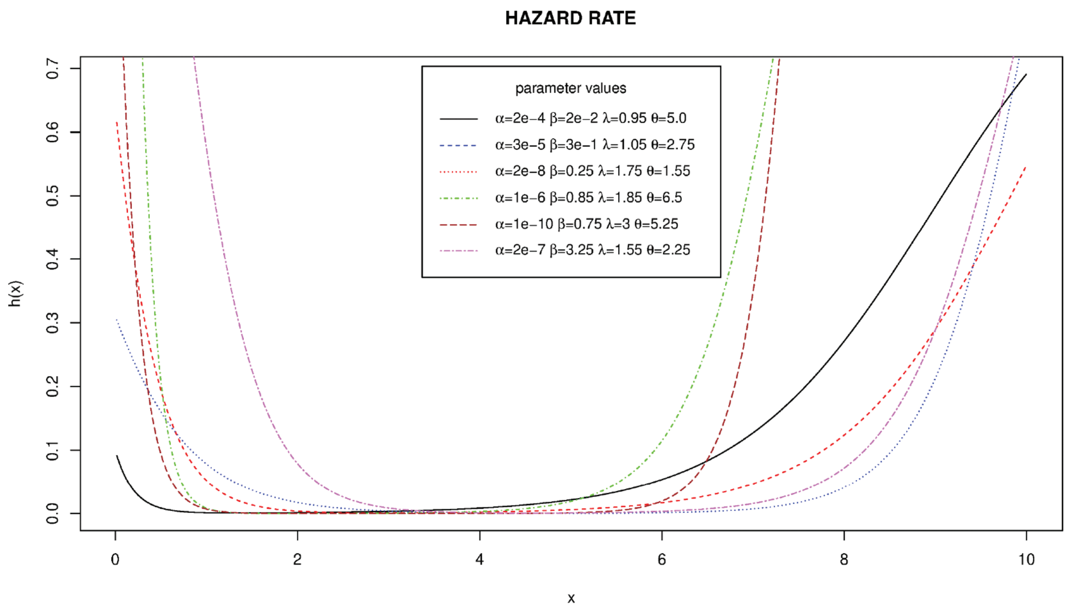


Figure 3. Hazard Rate for some parameter values derived of APD.

$$F(x) = 1 - \left(\frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)\left(1+\alpha e^{\lambda x}\right)}\right). \tag{4}$$

Figures 2 and 3 respectively show some representations of the PDF and the Hazard of the APD.

3. Measures of central tendency

In this section, some essential statistical metrics of the APD are presented.

3.1. Quantile

The $p-th$ quantile q_p denoted by x of APD based on the Eq.4, can be obtained as:

$$p = 1 - \left(\frac{(1+\alpha)\left(1+\beta e^{-\theta q}\right)}{(1+\beta)\left(1+\alpha e^{\lambda q}\right)}\right). \tag{5}$$

As can be seen, Eq.5 does not have an analytical solution in closed form; therefore, it must be estimated by numerical methods such as Newton-Raphson. Nevertheless, for some special cases, it is possible to get some solutions.

3.1.1. Special cases of quantile function

When $\lambda = 1,$ and $\theta = 1$:

$$q_x = \ln\left(\frac{\sqrt{-4\left(\alpha^2+\alpha-\frac{1}{4}p+\frac{1}{4}\right)(p-1)\beta^2 - 4(p-1)\left(\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}p\right)\beta + (\alpha+p)^2} + (-p+1)\beta - \alpha - p}{2\alpha(p-1)(1+\beta)}\right). \tag{6}$$

When $\alpha = 0,$ and $\lambda = 1$

$$q_x = -\frac{\ln\left(\frac{(-p+1)\beta-p}{\beta}\right)}{\theta}. \tag{7}$$

Other special cases can be obtained in a similar way.

3.2. The skewness and kurtosis

Skewness and kurtosis can be measured based on the quantile functions obtained in the subsection 3.1.1 and determine the shape of the APD. For this case, Bowley's expression Bowley (1920) and Moors expression Moors (1988) for skewness and kurtosis are used, respectively, and can be written as:

$$\Psi^* = \frac{Q(3/4) + Q(1/4) - 2 \cdot Q(1/2)}{Q(3/4) - Q(1/4)}. \tag{8}$$

$$\zeta^* = \frac{Q(7/8) + Q(3/8) - Q(5/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{9}$$

3.3. Mode

The mode for APD can be obtained by solving the equation:

$$f'(x) = 0.$$

$$-\frac{\left(\left(\alpha^2(\theta+\lambda)^2\left(e^{\lambda x}\right)^2 + 2\alpha\left(\theta^2+\theta\lambda-\frac{1}{2}\lambda^2\right)e^{\lambda x}+\theta^2\right)\beta e^{-\theta x} + \alpha\lambda^2 e^{\lambda x}\left(\alpha e^{\lambda x}-1\right)\right)(1+\alpha)}{\left(1+\alpha e^{\lambda x}\right)^3(1+\beta)} = 0. \tag{10}$$

The Eq. 10 does not have an analytical solution in general, so as in section 3.1, the mode of APD has to be estimated by numerical methods. The analytical solution can be estimated for some special cases, such as those presented in section 3.1.1.

4. Order statistics

The k-out-of-n is a popular representation in reliability engineering, which applies to redundant systems that handle information or perform critical processes (Kuo & Zuo, 2003). Let $x_1, x_2 \ldots x_n$ a random sample from the APD with PDF. Thus, the PDF of the *kth* order statistic x_k is given by:

$$f_{k:n}(x) = n\binom{n-1}{k-1}F(x)^{k-1}(1-F(x))^{n-k}f(x). \tag{11}$$

By taking Eq. 2 and Eq. 4 and replace them in Eq. 11, the following is obtained:

$$f_{k:n}(x) = n\binom{n-1}{k-1}\left[1-\left(\frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)\left(1+\alpha e^{\lambda x}\right)}\right)\right]^{k-1}. \tag{12}$$

$$\left[\frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})}\right]^{n-k}\cdot$$

$$\left(\frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}}+\frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta x}}\right)\cdot\frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})}.$$

After performing some mathematical operations, Eq. 12 can be reduced as:

$$f_{k:n}(x)=n\binom{n-1}{k-1}\cdot h(x)\cdot\sum_{i=0}^{\infty}\binom{k-1}{i}R(x)^{n+i+1-k}, \tag{13}$$

where $h(x)$ and $R(x)$ are the hazard function and reliability function of the APD. On the other hand, to obtain the smallest (x_1) and largest (x_n) statistical order, it is necessary to set $k=1$ and $k=n$, respectively.

5. Moments, MGF and incomplete moments

This section describes the equations of moments, and the MGF for the APD formulated in Eq. 2.

5.1. Moments

The *rth* row moment of APD is given by:

$$E(x^r)=\int_0^{\infty}rx^{r-1}R(x)dx.$$

$$=\frac{r(1+\alpha)}{1+\beta}\int_0^{\infty}x^{r-1}\cdot\left(1+\beta e^{-\theta x}\right)\cdot\left(1+\alpha e^{\lambda x}\right)^{-1}dx$$

By binomial and Taylor series expansion, we know that:

$$(1+ae^{-bx})=\sum_{i=0}^{\infty}a^{i}e^{-bix},$$

$$e^{-bix}=\sum_{j=0}^{\infty}\frac{-b^{j}i^{j}}{\Gamma(j+1)}.$$

Therefore, applying the series expansions discussed above and applying algebraic reduction, the *rth* Moment is written as:

$$E(x^r)=\frac{r(1+\alpha)}{1+\beta}\sum_{i,j,\ell=0}^{\infty}\frac{\beta^{i}(-\theta i)^{j}(-\alpha)^{\ell}}{i!}\Gamma(i+j+\ell), \tag{14}$$

where $\Gamma(.)$ is the gamma function.

5.2. MGF

The MGF of X is given by:

$$M_x(t)=E(e^{tx})$$

$$M_x(t) = \int_0^\infty e^{tx} f(x) dx.$$

By series expansion, it is known that:

$$e^{tx} = \sum_{u=0}^{\infty} \frac{(tx)^u}{u!}.$$

Therefore, $M_x(t)$ can be rewritten as:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^u}{u!} \int_0^\infty x^u f(x) dx,$$

$$= \sum_{u=0}^{\infty} \frac{t^u}{u!} \int_0^\infty r x^{r-1} R(x) dx.$$

Finally, by substituting the obtained in Eq. 14, the MGF can be written as:

$$M_x(t) = \frac{r(1+\alpha)}{1+\beta} \sum_{i,j,\ell,u=0}^{\infty} \frac{\beta^i (-\theta i)^j (-\alpha)^\ell t^u}{i! u!} \Gamma(i+j+\ell). \tag{15}$$

The characteristic function of the APD can be obtained with the same procedure followed by MGF. Just, replace the expansion series for $e^{itx} = \sum_{u=0}^{\infty} \frac{(itx)^u}{u!}$

5.3. *Incomplete moments*

One aspect to consider within lifetime models is the incomplete moments related to the Bonferroni and Lorenz curves. The incomplete moments are defined as:

$$m_r(t) = \int_0^t x^r f(x) dx.$$

$$= \int_0^t x^r \cdot \left[\left(\frac{\alpha \lambda e^{\lambda x}}{1+\alpha e^{\lambda x}} + \frac{\beta \theta e^{-\theta x}}{1+\beta e^{-\theta x}} \right) \cdot \frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})} \right] dx,$$

$$= \frac{1+\alpha}{1+\beta} \cdot \int_0^t x^r \cdot \frac{\beta\left[\alpha(1+\theta)e^{\lambda x} + \theta\right] e^{-\theta x} + \alpha e^{\lambda x}}{(1+\alpha e^{\lambda x})^2} dx.$$

Applying series expansion

$$= \frac{1+\alpha}{1+\beta} \cdot \sum_{i,j=0}^{\infty} \frac{(2\alpha)^i (\lambda i)^j}{\Gamma(j+1)} \cdot \int_0^t x^{r+j} \cdot \left\{ \beta\left[\alpha(1+\theta)e^{\lambda x} + \theta\right] e^{-\theta x} + \alpha \lambda e^{\lambda x} \right\} dx.$$

By solving the above integral, we get

$$m_r(t) = \frac{1+\alpha}{1+\beta} \cdot \sum_{i,j=0}^{\infty} \frac{(2\alpha)^i (\lambda i)^j}{\Gamma(j+1)} t^{jr} \left\{ \frac{\alpha\beta(\lambda+\theta)[t(\theta-\lambda)]^{-j-r} \gamma_1}{\lambda - \theta} + \alpha(-\lambda t)^{-j-r} \gamma_2 - \beta(\theta t)^{-j-r} \gamma_3 \right\}, \tag{16}$$

where

$$\gamma_1 = \Gamma(j+r+1, (\theta-\lambda)t).$$

$$\gamma_2 = \Gamma(j + r + 1, -\lambda t).$$

$$\gamma_3 = \Gamma(j + r + 1, -\theta t).$$

6. Residual and reversed lifetime function

The residual and reversed lifetime functions play an essential role in reliability analysis to understand a device's lifetime and failure time under an operational environment. The residual life can be defined by the conditional random variable $R_t = X - t|X > t, t \geq 0$ and represents the period from t until the time of failure of the device. Mathematically, the residual life can be written as:

$$S_{R(t)}X = \frac{R(x+t)}{R(t)}. \tag{17}$$

Based on Eq. 17 and by taking the Eq. 3, the residual lifetime of APD is given by:

$$S_{R(t)}X = \frac{\left(1 + \beta e^{-\theta(x+t)}\right)\left(1 + \alpha e^{\lambda x}\right)}{\left(1 + \alpha e^{\lambda(x+t)}\right)\left(1 + \beta e^{-\theta x}\right)}. \tag{18}$$

The reversed residual lifetime denoted by $F_t = t - X|X \leq t$ represents the time elapsed from the failure of a component in the device given its life $\leq t$. The reversed residual life can be written as:

$$S_{F(t)}X = \frac{F(t-x)}{F(t)} \tag{19}$$

Based on Eq. 19 and by taking the Eq. 4, the reversed residual lifetime of APD is given by:

$$S_{F(t)}X = -\frac{\left(\beta(1+\alpha)\mathrm{e}^{-\theta(t-x)} - \alpha(1+\beta)\mathrm{e}^{l(t-x)} + \alpha - \beta\right)\left(1 + \alpha\,\mathrm{e}^{\lambda x}\right)}{\left(1 + \alpha\,\mathrm{e}^{\lambda(t-x)}\right)\left(-\beta(1+\alpha)\mathrm{e}^{-\theta x} + a(1+\beta)\mathrm{e}^{\lambda x} - \alpha + \beta\right)}. \tag{20}$$

7. Entropy

Entropy is used in reliability to measure the uncertainty of the information obtained. In the case of the APD, the entropy of Rényi (Rényi, 1961) has been investigated, which is defined as:

$$I_R(\delta) = \frac{1}{1-\delta} \cdot log \int_0^{\infty} f(x)^{\delta} dx. \tag{21}$$

By taking the Eq.2 and substituting in the Eq. 21, the Rényi Entropy for the APD can be calculated as follows:

$$I_R(\delta) = \frac{1}{1-\delta} \cdot log \int_0^{\infty} \left\{ \left(\frac{\alpha\lambda e^{\lambda x}}{1 + \alpha e^{\lambda x}} + \frac{\beta\theta e^{-\theta x}}{1 + \beta e^{-\theta x}} \right) \cdot \frac{(1+\alpha)\left(1 + \beta e^{-\theta x}\right)}{(1+\beta)\left(1 + \alpha e^{\lambda x}\right)} \right\}^{\delta}.$$

In the first instance, it is necessary to develop $f(x)^{\delta}$. For this, we will use binomial and Taylor series and we know that:

$$(a+b)^n = \sum_{i=0}^{n} = a^{n-i} b^i.$$

Therefore, by applying the previous series expansion, and in turn, those used in the section 5, the $f(x)^{\delta}$ is reduced to the following term:

$$f(x)^{\delta} = \left(\frac{1+\alpha}{1+\beta}\right)^{\delta} \sum_{i=0}^{\delta} \sum_{j,\ell,p,u=0}^{\infty} \binom{\delta}{i} (\delta - i)^{(j+u)} (\alpha\lambda)^{(j+\ell)} (\theta i)^{p} \beta^{(p+u)} (i - 2\delta)^{\ell}$$

$$e^{(\lambda(j+\ell)-\theta(p+u))x}. \tag{22}$$

Taking the result obtained in Eq. 22, the Renyi entropy for the APD is:

$$I_R(\delta) = \left(\frac{1+\alpha}{1+\beta}\right)^{\delta} \cdot \sum_{i=0}^{\delta} \sum_{j,\ell,p,u=0}^{\infty} \binom{\delta}{i} (\delta - i)^{(j+u)} (\alpha\lambda)^{(j+\ell)} (\theta i)^{p} \beta^{(p+u)} (i - 2\delta)^{\ell}$$

$$ln\left[-\frac{1}{\lambda(j+\ell) - \theta(p+u)}\right]. \tag{23}$$

8. Time-varying and ALT modeling for APD

In this section, we consider two models that may be useful for reliability engineering practitioners to determine the behavior of devices under real operating situations. Firstly, we establish an approach to model the APD in a time-varying scenario. As the second model, we propose a life-stress relationship where the APD and the Inverse Power Law (IPL) are combined to describe devices' behavior through an ALT.

8.1. APD time-varying model

Let $x(u)$ be a function that describes the behavior of the stress supplied to the device under analysis during a time t. The PDF of the APD described in Eq.2 can be described as:

$$f(t, x(t)) = \left(\frac{\alpha\lambda e^{\lambda\gamma}}{1+\alpha e^{\lambda\gamma}} + \frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta\gamma}}\right) \cdot \frac{(1+\alpha)\left(1+\beta e^{-\theta\gamma}\right)}{(1+\beta)(1+\alpha e^{\lambda\gamma})}, \tag{24}$$

where $\gamma = \int_0^t x(u)du$ and $x(u) > 0$.

Eq.24 is helpful in cases where the analysis is focused on obtaining the behavior of the device in real operating environments. An example of this is presented in ELD, where the voltage variations in the supply line can reduce the average lifetime and the device's performance over time.

8.2. APD-IPL modeling

The following proposed model is valid when accelerated experimentation data are obtained. ALTs are one of the most widely used techniques in the field of reliability to quickly obtain failure times by accelerating some physical variable with which the device is affected.

The models that describe the behavior of the devices under an ALT are built through two elements. One is the statistical distribution that describes the device's behavior over time. The statistical distribution that will be part of the life-stress model is the PDF of the APD presented in Eq. 2, with the behavior of the data obtained from the ALT will be represented. The second is a stress relationship that describes the device under a physical variable. The relation selected for this case is the IPL, defined as an empirical model where the life of a device is linked to a certain level of an accelerated variable, which directly affects the device's life. Generally, the IPL is used in reliability analysis to determine the behavior of the ED under an accelerated voltage variable, which directly affects the lifetime of the ED. Mathematically, the IPL is defined as:

$$\Lambda = \frac{1}{\varsigma\omega^{\rho}}, \tag{25}$$

where $\varsigma > 0$ represents the physical characteristics of the internal components with which the device under analysis is built. $\omega > 0$ represents the stress level applied during the ALT. For the case of determining reliability in ED, the most common stress is voltage. Parameter $\rho > 0$ measures the effects of stress on the analyzed device. The device's lifetime is considerably affected if the value of ρ is substantial.

Taking Eq.25 and substituting it in the shape parameters of the APD proposed in Eq.2, the PDF of the APD-IPL stress-life relationship can be written as:

$$f_{APD-IPL}(x) = \frac{\varsigma\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)(\varsigma\omega^{\rho}+1)\omega\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)^{\rho\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)}\left(((\theta+\lambda)e^{\lambda x}+\theta\varsigma\omega^{\rho})e^{-\theta x}+\lambda\varsigma\omega^{\rho}e^{\lambda x}\right)}{\varsigma^{2}(\omega^{\rho})^{2}(\varsigma\omega^{\rho}+e^{\lambda x})\left(\varsigma\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)\omega\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)^{\rho\left(1+\frac{e^{\lambda x}}{\varsigma\omega^{\rho}}\right)}+1\right)}. \tag{26}$$

9. Maximum Likelihood Estimator (MLE)

An essential aspect of the APD is the estimation of the parameters $(\alpha, \beta, \lambda, \theta)$; for this, the MLE is used.

Let $X_1, X_2 \ldots X_m$ a random sample from the lifetime distribution with PDF $f(x)$ and based on sample of size m. Then the likelihood function of Eq. 2 is written as follows:

$$L = \prod_{i=1}^{m}\left\{\left(\frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}}+\frac{\beta\theta e^{-\theta x}}{1+\beta e^{-\theta x}}\right)\cdot\frac{(1+\alpha)\left(1+\beta e^{-\theta x}\right)}{(1+\beta)(1+\alpha e^{\lambda x})}\right\}.$$

By taking the logarithm of the above equation, the log-likelihood function (Λ) becomes:

$$\Lambda = \sum_{i=1}^{m}\left[\ln\left(\frac{\alpha\lambda e^{\lambda x_i}}{1+\alpha e^{\lambda x_i}}+\frac{\beta\theta e^{-\theta x_i}}{1+\beta e^{-\theta x_i}}\right)\right] + m\ln(1+\alpha) - m\ln(1-\beta)$$

$$+\sum_{i=1}^{m}\ln\left[1+\beta e^{-\theta x_i}\right] - \sum_{i=1}^{m}\ln\left[1+\alpha e^{\lambda x_i}\right]. \tag{27}$$

By taking the first partial derivative of Eq. 27, the estimation for each parameter α, β, λ, and θ can be calculated as follows:

$$\frac{\partial\Lambda}{\partial\alpha} = \frac{m}{1+\alpha}+\sum_{i=1}^{m}\left(\frac{\lambda e^{\lambda x_i}}{(1+\alpha e^{\lambda x_i})^2 h(x_i)}-\frac{e^{\lambda x_i}}{1+\alpha e^{\lambda x_i}}\right), \tag{28}$$

$$\frac{\partial\Lambda}{\partial\beta} = \frac{m}{1-\beta}+\sum_{i=1}^{m}\left(\frac{\theta e^{-\theta x_i}}{(1+\beta e^{-\theta x_i})^2 h(x_i)}+\frac{e^{-\theta x_i}}{1+\beta e^{-\theta x_i}}\right), \tag{29}$$

$$\frac{\partial\Lambda}{\partial\lambda}=\sum_{i=1}^{m}\left(\frac{\alpha\,e^{\lambda x_i}\left(\alpha\,e^{\lambda x_i}+\lambda x_i+1\right)}{\left(1+\alpha\,e^{\lambda x_i}\right)^2h(x_i)}-\frac{\alpha x_i e^{\lambda x_i}}{1+\alpha\,e^{\lambda x_i}}\right), \tag{30}$$

$$\frac{\partial\Lambda}{\partial\theta}=\sum_{i=1}^{m}\left(\frac{\beta\,e^{-\theta x_i}\left(\beta\,e^{-\theta x_i}-\theta x_i+1\right)}{\left(1+\beta\,e^{-\theta x_i}\right)^2h(x_i)}-\frac{\beta x_i e^{-\theta x_i}}{1+\beta\,e^{-\theta x_i}}\right), \tag{31}$$

where $h(x_i)$ is the hazard function of the APD presented in Eq.1.
The Fisher information matrix based on Eq.28 to Eq.31 is given by:

$$J(\delta)=-\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} & I_{\alpha\theta} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} & I_{\beta\theta} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} & I_{\lambda\theta} \\ I_{\theta\alpha} & I_{\theta\beta} & I_{\theta\lambda} & I_{\theta\theta} \end{bmatrix} \tag{32}$$

The elements of the fisher matrix are shown in Appendix A.
To obtain the confidence intervals, we based on the asymptotic normality of MLEs. The $100(1-\zeta)\%$ confidence intervals for $\alpha, \beta, \lambda, and\theta$ are given by:

$$\left(\hat{\alpha}\pm z_{1-\rho/2}\sqrt{var(\hat{\alpha})}\right). \tag{33}$$

$$\left(\hat{\beta}\pm z_{1-\rho/2}\sqrt{var\left(\hat{\beta}\right)}\right). \tag{34}$$

$$\left(\hat{\lambda}\pm z_{1-\rho/2}\sqrt{var\left(\hat{\lambda}\right)}\right). \tag{35}$$

$$\left(\hat{\theta}\pm z_{1-\rho/2}\sqrt{var\left(\hat{\theta}\right)}\right), \tag{36}$$

where, $z_{1-\rho/2}$ is the upper $(\rho/2)$ percentile of the standard normal distribution.

10. Cases of study

In this section, the APD is tested in three case studies to highlight the importance of the proposed distribution in the reliability analysis. For this, the APD is compared with other distributions that can describe the behavior of the bathtub curve; the list of the compared distributions is shown in Table 1. The MLE and the optimization method Newton-Raphson programmed in RStudio were employed to estimate the parameters. The AIC, BIC, K-S, AD, CVM, and P-values were calculated to derive the conclusions for each distribution.

Table 1. Bathtub shape distributions.

Model	$h(x)$
Perks Distribution (PD)	$\alpha\lambda e^{\lambda x}\left(1+\alpha e^{\lambda x}\right)^{-1}$
Additive Weibull (AWD)	$\alpha\lambda x^{\lambda-1}+\beta\theta x^{\theta-1}$
Alpha Power Transformation Weibull (APTW)	$\ln(\alpha)\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(\alpha^{e^{-\lambda x^{\beta}}}\right)^{-1}$
Modified Weibull Extension (MWE)	$\lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}e^{\left(\frac{x}{\alpha}\right)^{\beta}}$
S-Z modified Weibull (SZMW)	$\alpha+\beta\lambda x^{\lambda-1}$
Exponentiated Perks Distribution (EPD)	$\frac{\beta\lambda\alpha^{\beta}(1+\alpha)e^{\lambda x}\left(e^{\lambda x}-1\right)^{\beta-1}}{\left(1+\alpha e^{\lambda x}\right)\left[\left(1+\alpha e^{\lambda x}\right)^{\beta}-\alpha^{\beta}\left(e^{\lambda x}-1\right)^{\beta}\right]}$

Table 2. Aarset Data of 50 Devices for the case of Study 1.

DATA									
0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

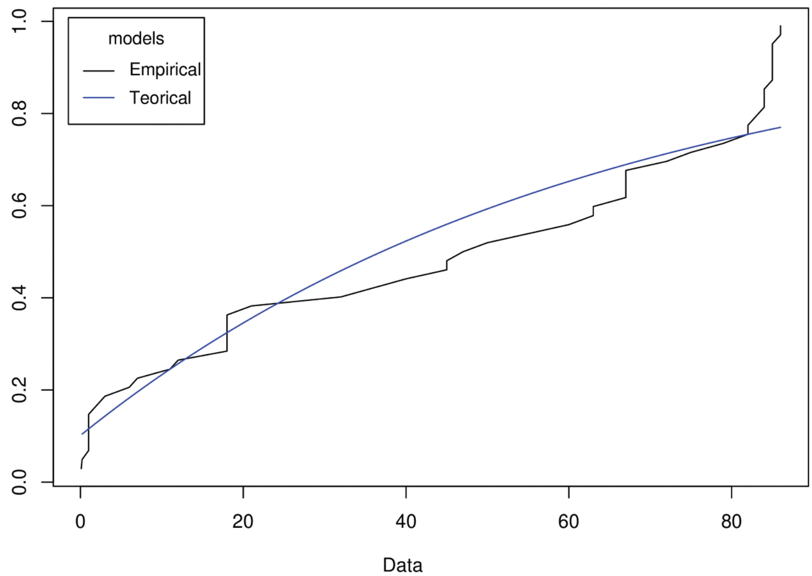


Figure 4. TTT-Plot for the Aarset data.

10.1. Case of study 1

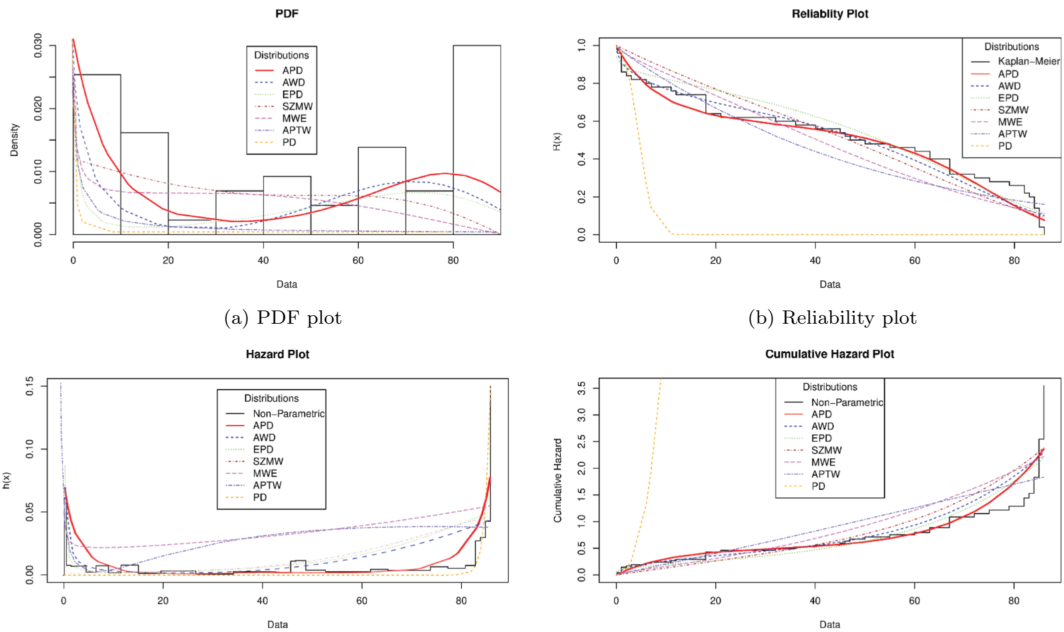
For this case study, we will use the data obtained by Aarset (1987), which consists of the failure times of 50 devices. Data are presented in Table 2 and the TTT (fitted and empirical) plot is presented in Figure 4.

Table 3 shows the results of the parameters of each of the models compared against the APD, their respective MLE's, standard errors, the AIC, BIC, K-S, AD, CVM, and the P-value. The conclusions of Table 3 show that the APD has a better performance in connection with AIC, BIC, K-, AD, and CVM concerning the models that describe the failure rate in the form of a bathtub curve presented in Table 1.

In Figure 5, the reliability plots are shown for each distribution under analysis for case study 1. In Figure 5a, the shape of the PDF for the proposed distribution (APD) is shown; it has a better fit to the histogram of the Aarset data compared to the models established in Table 1. In Figure 5b the reliability graph is appreciated, where the models under analysis are compared with the empirical reliability, the results of this Figure 5b show that the APD model has a better approach to the empirical reliability for the data established in Table 2. This approach implies that the APD model can better describe the behavior of the operating lifetime of the device under analysis. Figure 5c shows the shape of the failure times for the

Table 3. Estimated Values, standard errors in parentheses and Statistics metrics, *P*-Value in parentheses for the Case of Study 1.

	Parameters				Statistics					
Model	α	β	λ	θ	Loglik	AIC	BIC	K-S	AD	CVM
APD	$1.748e-3(6.538e-4)$	0.695(0.242)	$9.049e-2(4.294e-3)$	$9.683e-2(3.609e-2)$	− 219.224	446.449	454.096	0.080(0.832)	0.221(0.855)	0.088(0.824)
AWD	$8.451e-9(1.498e-9)$	$9.121e-2(3.821e-2)$	$4.279(4.912e-2)$	$0.456(9.831e-2)$	− 221.441	450.874	458.974	0.112(0.451)	0.571(0.466)	0.271(0.449)
EPD	$9.779e-5(3.828e-4)$	$2.024e-1(8.015e-2)$	$1.100e-1(4.207e-2)$	–	− 226.057	458.114	463.787	0.125(0.441)	0.556(0.435)	0.246(0.446)
SZMW	$0.013(3.00e-3)$	$3.809e-9(6.807e-10)$	4.405(0.143)	–	− 229.412	464.823	470.564	0.147(0.201)	1.214(0.212)	0.436(0.207)
MWE	13.736(5.088)	$0.588(7.814e-2)$	$9.014e-3(3.141e-2)$	–	− 231.657	469.296	475.032	0.155(0.175)	1.424(0.165)	0.491(0.170)
APTW	4.355(0.889)	$5.728e-2(4.008e-2)$	0.840(0.142)	–	− 239.622	485.245	491.884	0.187(0.154)	1.697(0.159)	0.674(0.157)
PD	0.6167(0.1283)	–	$3.269e-2(6.98e-3)$	–	− 239.553	483.107	489.224	0.214(0.114)	2.146(0.121)	0.874(0.117)



(a) PDF plot (b) Reliability plot

(c) Hazard plot (d) Cumulative Hazard plot

Figure 5. Reliability Plots for the Aarset Data of Case of Study 1.

Table 4. Meeker and Escobar Data of 30 Devices for the case of Study 2.

DATA									
275	13	147	23	181	30	65	10	300	173
106	300	300	212	300	300	300	2	261	293
88	247	28	143	300	23	300	80	245	266

data presented in Table 2. This graph shows that the APD model approximates the bathtub shape better than the other models for this study case.

On the other hand, it is observed that concerning the non-parametric curve, the curve drawn by the APD is adjusted in a more significant number of points, so the description of the faults under the APD may be better. Finally, the cumulative hazard plot is shown in Figure 5d. For this case, the APD also offers a better fit compared to the models in Table 1 and a good approximation of the non-parametric curve.

10.2. Case of study 2

This case study is based on the data presented by Meeker et al. (2021). The data characteristics consist of 30 devices from a large system field-tracking study. Table 4 presents the data from this case study. In Figure 6, the TTT plot is shown, which empirically and theoretically defines the behavior of the failure times of the data presented in Table 4.

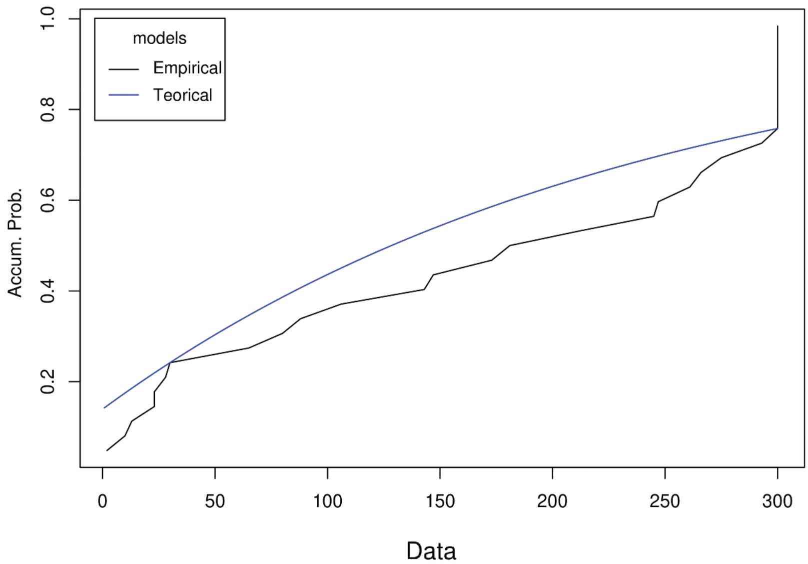


Figure 6. TTT-Plot for the Meeker & Escobar data Case Study 2.

Table 5 shows that the APD can be a better model for representing the failure times presented in Table 4. This conclusion can be derived from the AIC, the BIC, K-S, AD, CVM and P-value criteria, where the APD has good results indices of all distributions listed in Table 1.

In Figure 7, the reliability graphs of the behavior of the data for the case of study 2 are appreciated. Figure 7a shows the behavior of the PDF of case 2. In this case, it can be seen that the APD has a shape where not all the histogram points. Nevertheless, the shape of the APD has a better adjustment than the other models. Figure 7b shows the behavior of the reliability of the case 2 data. The evidence shows that the APD touches the line drawn by the Kaplan–Meier method in a more significant number of points, which indicates that the APD will more closely describe the behavior of the device under analysis.

On the other hand, we can observe that the other models under analysis are far from the line of empirical reliability; this can directly affect the calculation of the warranty times and the MTTF in the device. Figure 7c shows the behavior of the failure times of the data from case study 2 concerning each of the models under analysis. As can be seen, the APD shows behavior that closely resembles the shape of a bathtub curve and touches more points in the non-parametric risk function. Likewise, the other distributions under analysis show behavior that does not resemble a bathtub curve, which affects the estimation of the device's behavior under analysis. Figure 7d shows the behavior of the cumulative hazard plot; as can be seen, the APD fits better with the non-parametric curve established for the data of case study 2. The above indicates that the APD is a good choice for describing failure times compared to the other models under review.

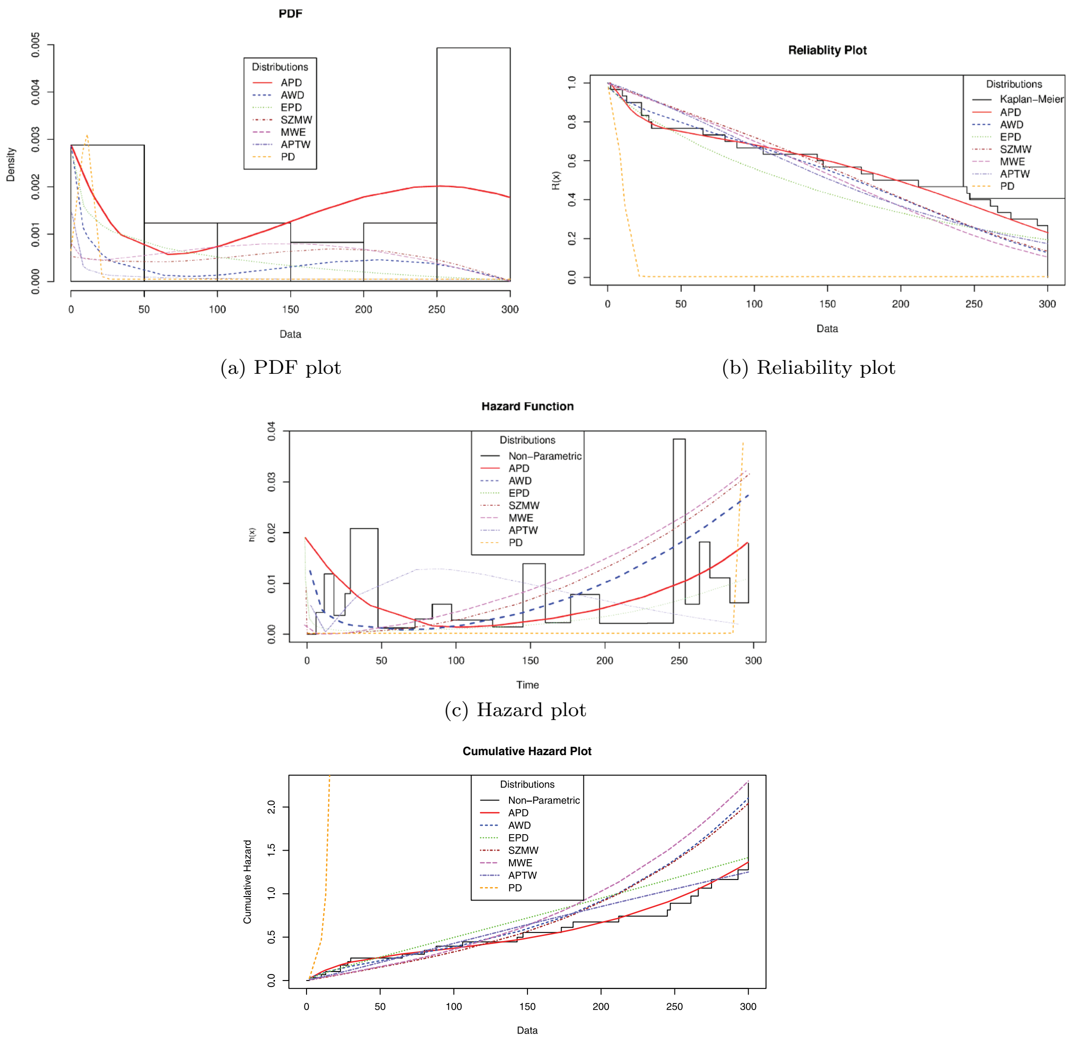
10.3. Case of study 3

This case study is represented by the failure times of 18 ELD presented by Wang (2000); the failure times mentioned above can be seen in Table 6. Figure 8 shows the behavior (empirical and theoretical) of the TTT plot for the data presented in Table 6.

Estimated values for Case of Study 3, are presented in Table 7.

Table 5. Estimated Values, standard errors in parentheses and Statistics metrics, *P*-Value in parentheses for the Case of Study 2.

	Parameters				Statistics					
Model	α	β	λ	θ	Loglik	AIC	BIC	K-S	AD	CVM
APD	$5.583e-3(2.521e-3)$	0.447(0.205)	$2.055e-2(1.4e-3)$	$1.669e-2(8.080e-3)$	− 175.606	359.213	364.818	0.216(0.651)	0.541(0.675)	0.302(0.662)
AWD	$6.914e-9(9.868e-10)$	$1.821e-2(1.745e-2)$	$3.351(5.412e-2)$	0.645(0.194)	− 177.315	362.630	370.278	0.165(0.465)	0.701(0.478)	0.445(0.469)
EPD	1.787(0.147)	0.984(0.283)	$5.920e-3(1.644e-3)$	–	− 184.629	375.258	380.994	0.170(0.477)	0.735(0.455)	0.491(0.463)
SZMW	$2.981e-3(2.00e-2)$	$8.847e-9(2.944e-9)$	$3.275(6.21e-2)$	–	− 178.447	362.894	368.630	0.174(0.344)	0.763(0.372)	0.521(0.332)
MWE	89.784(4.488)	$0.798(9.914e-2)$	$2.014e-3(6.014e-4)$	–	− 179.998	365.996	371.732	0.178(0.297)	0.824(0.309)	0.598(0.315)
APTW	3.792(0.433)	$3.698e-3(5.417e-3)$	1.128(0.241)	–	− 183.672	373.345	379.080	0.188(0.214)	1.025(0.209)	0.664(0.228)
PD	0.257(0.220)	–	$1.067e-2(2.528e-3)$	–	− 182.764	369.529	373.352	0.194(0.197)	1.326(0.182)	0.782(0.200)



(a) PDF plot (b) Reliability plot

(c) Hazard plot

(d) Cumulative Hazard plot

Figure 7. Reliability Plots for the Aarset Data of Case of Study 1.

Table 6. Wang Data of 18 ELD for the case of Study 3.

Data								
5	11	21	31	46	75	98	122	145
165	196	224	245	293	321	330	350	420

Table 7 shows the data from case study 3. The APD is a better option for describing failure times based on the AIC, BIC, K-S, AD CVM, and P-value criteria, in which the APD has the best indices for all distributions under analysis.

Figure 9 shows the graphs of the reliability behavior for the data of case study 3. Figure 9a shows the behavior of the PDF for all the distributions analyzed with the data presented in Table 6. As can be seen, the PDF of the APD touches on more points than the histogram of the data representation in Table 6. Figure 9b shows the reliability graph for the data from case study 3. The results of Figure 9b

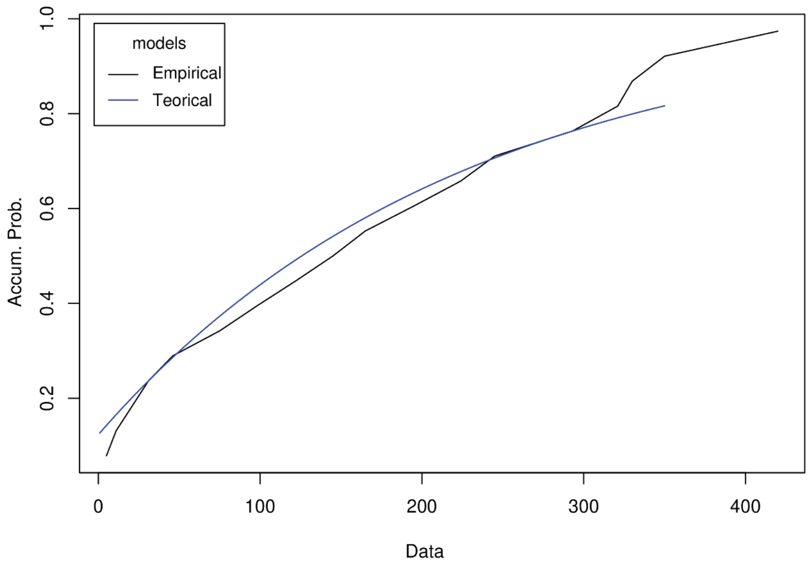
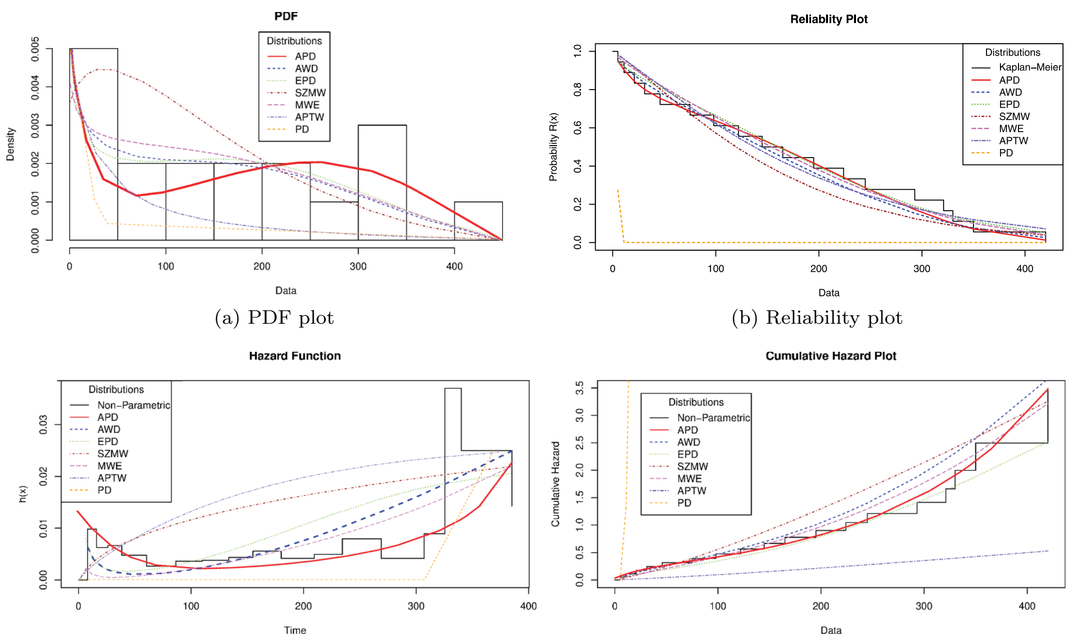


Figure 8. TTT-Plot for the Wang data Case Study 3.

show that the APD reliability fits slightly better than the reliability line established by the Kaplan–Meier method. Figure 9c shows the behavior of the failure times for the case of study 3. From this figure, it is noted that the APD has similar behavior to the bathtub curve. In contrast, the other models proposed in the analysis do not have a non-monotonic behavior. Only the AWD model comes slightly closer to behavior in the shape of a bathtub curve. Finally, as shown in Figure 9d, the APD fits the non-parametric representation better, which is taken as the basis for understanding the behavior of the data.



(a) PDF plot

(b) Reliability plot

(c) Hazard plot

(d) Cumulative Hazard plot

Figure 9. Reliability Plots for the Aarset Data of Case of Study 1.

Table 7. Estimated Values, standard errors in parentheses and Statistics metrics, *P*-Value in parentheses for the Case of Study 3.

	Parameters				Statistics					
Model	α	β	λ	θ	Loglik	AIC	BIC	K-S	AD	CVM
APD	$0.170(3.868e-2)$	$0.221(6.144e-2)$	$7.362e-3(5.1726e-3)$	$5.532e-2(2.962e-2)$	−108.373	224.746	228.308	0.081(0.948)	0.117(0.953)	$1.115e-2(0.939)$
AWD	$4.366e-7(4.147e-6)$	$1.521e-2(1.745e-2)$	$2.578(1.457)$	$0.714(0.301)$	−109.451	226.902	230.463	0.101(0.801)	0.257(0.811)	$3.336e-2(0.800)$
EPD	$7.152e-2(1.254e-2)$	$0.540(2.987e-2)$	$1.187e-2(5.061e-3)$	–	−109.978	225.956	228.627	0.145(0.172)	0.335(0.165)	$5.232e-2(0.178)$
SZMW	$2.321e-3(7.154e-3)$	$6.287e-4(4.974e-3)$	$1.357(0.765)$	–	−110.447	226.894	229.565	0.155(0.701)	0.387(0.689)	$5.571e-2(0.720)$
MWE	$13.472(9.124)$	$0.745(0.145)$	$2.541e-3(7.114e-4)$	–	−111.478	228.956	231.627	0.161(0.672)	0.411(0.655)	$6.264e-2(0.682)$
APTW	$2.541(0.355)$	$5.558e-3(1.099e-2)$	$1.044(0.306)$	–	−110.306	226.613	229.683	0.291(0.204)	0.997(0.195)	0.325(0.218)
PD	$1.162(0.107)$	–	$7.752e-3(9.821e-3)$	–	−113.225	230.450	232.230	0.341(0.111)	1.549(0.101)	0.398(0.118)

As a conclusion of this case study, we can establish the flexibility of the APD since, within the description of the failure times presented in Figure 9c, it was the one that obtained the best behavior and representation of the bathtub curve. The above indicates that the APD can be used for different data types from reliability tests.

10.4. Discussion of results

The results show that the APD may be the best option to describe the behavior of failure times in each of the analyzed case studies. In all cases, the APD showed behaviors similar to the bathtub curve. This advantage offered by the APD to practitioners of reliability analysis allows them to calculate maintenance times more precise and closer to the reality of the device under analysis, the MTTF, and the development of a plan to improve product quality. Therefore, based on the evidence shown in the case studies, practitioners could be inclined to select the APD concerning some classical or hybrid distribution described in the literature review since, in some cases, the distributions are put under analysis despite having the bathtub curve property. These distributions did not represent such behavior in the hazard plot, so the information was biased, which means that the behavior of the data was not completely described as in the case of the bathtub curve. From the manufacturing point of view, the loss of information mentioned above can be vital for the increase or not of manufacturing costs; this is due to incorrect planning of the number of devices that fail before fulfilling the warranty. Therefore, the distribution with which the reliability analysis is carried out must be the most faithful to the design and characteristics of the product.

On the other hand, the implementation of APD is similar to any classical distribution used in reliability analysis. Therefore, adjusting the failure times to the APD will depend on the design of the experiment with which the failure times are obtained, whether accelerated or not, and on the device's physical characteristics under analysis. In turn, given the flexibility of the APD parameters, it allows reliability engineering practitioners to test any device, whether or not the failure times have non-monotonic behavior. This flexibility makes the APD a quite versatile distribution when representing the behavior of a device or system.

11. Conclusion and future scope

This paper presented the APD for reliability analysis, which has two-shape and two-scale parameters. The proposed distribution was based on the sum of the positive and negative risk functions of the Perks distribution. One of the most notable characteristics of the APD is the flexibility to represent failure times in a non-monotonic way. Statistical properties such as measures of central tendency, order statistics, and the function of moments were developed. On the other hand, to make APD more attractive to reliability engineering practitioners, time-varying modeling was presented; the APD in time-varying can be used to determine the reliability of devices closer to real conditions. At the same time, the APD-IPL modeling is presented and is useful when the experimental data comes from an ALT. The parameters of APD were calculated via MLE; for this, an R code was developed. Three case studies were analyzed to verify the proposed distribution, and the results were compared against other distributions with similar properties to the APD. The case studies focused on failure times with bathtub curve characteristics. The results showed that the APD is a competitive model to represent failure times with a bathtub curve. Finally, the APD proved to be a flexible model since in the three study cases presented; it was the one that obtained the best bathtub curve representations in comparison with the models presented in Table 1.

A Bayesian analysis may be considered for future research to estimate the APD parameters. In turn, given the flexibility of the APD, it can be mixed with other distributions to obtain a more robust model. Finally, developing other life-stress models can be considered for analyzing different stress variables used in ALT.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix

Appendix A. OBSERVED FISHER MATRIX ELEMENTS

In this section, the elements of the Fisher Matrix presented in Eq.32 are detailed.

$$a = \alpha^2\left(x_i h(x_i)^2 + (\lambda x_i - 1)h(x_i) + \lambda\right),$$

$$b = \alpha\left(\lambda^2 x_i + 2x_i h(x_i)^2 + \lambda - 2h(x_i)\right)e^{\lambda x_i},$$

$$c = -2\theta\left(1+\beta e^{-\theta x_i}\right)e^{-2\theta x_i} + 2\beta\theta e^{-3\theta x_i}.$$

$$I_{\alpha\alpha} = \sum_{i=1}^{m}\left(-\frac{2\lambda e^{2\lambda x_i}}{(1+\alpha e^{\lambda x_i})^3 h(x_i)} - \frac{\lambda^2 e^{2\lambda x_i}}{(1+\alpha e^{\lambda x_i})^4 h(x_i)^2} + \frac{e^{2\lambda x_i}}{(1+\alpha e^{\lambda x_i})^2}\right) - \frac{m}{(1+\alpha)^2}.$$

$$I_{\alpha\beta} = -\sum_{i=1}^{m}\left(\frac{\lambda e^{x_i(\lambda-\theta)}\theta}{(1+\alpha e^{\lambda x_i})^2(1+\beta e^{-\theta x_i})^2 h(x_i)^2}\right).$$

$$I_{\alpha\lambda} = -\sum_{i=1}^{m}\left(\frac{e^{\lambda x_i}\left(a+b-\lambda x_i h(x_i) + x_i h(x_i)^2 - h(x_i)\right)}{(1+\alpha e^{\lambda x_i})^4 h(x_i)^2}\right).$$

$$I_{\alpha\theta} = -\lambda\beta\sum_{i=1}^{m}\left(\frac{e^{x_i(\lambda-\theta)}\left(\beta e^{-\theta x_i} - \theta x_i + 1\right)}{(1+\alpha e^{\lambda x_i})^2(1+\beta e^{-\theta x_i})^2 h(x_i)^2}\right).$$

$$I_{\beta\alpha} = I_{\alpha\beta}$$

$$I_{\beta\beta} = \frac{m}{(1-\beta)^2} + \sum_{i=1}^{m}\left(\frac{c}{(1+\beta e^{-\theta x_i})^3 h(x_i)} - \frac{\theta^2 e^{-2\theta x_i}}{(1+\beta e^{-\theta x_i})^4 h(x_i)^2} - \frac{e^{-2\theta x_i}}{(1+\beta e^{-\theta x_i})^2}\right).$$

$$I_{\beta\lambda} = \sum_{i=1}^{m}\left(-\frac{e^{x_i(\lambda-\theta)}\theta\alpha\left(\alpha e^{\lambda x_i} + \lambda x_i + 1\right)}{(1+\beta e^{-\theta x_i})^2(1+\alpha e^{\lambda x_i})^2 h(x_i)^2}\right).$$

$$I_{\beta\theta} = \sum_{i=1}^{m}\left(\frac{e^{-\theta x_i}\left(\beta(\theta x_i+1)e^{-\theta x_i} - \theta x_i + 1\right)}{(1+\beta e^{-\theta x_i})^3 h(x_i)} - \frac{\theta e^{-2\theta x_i}\beta\left(\beta e^{-\theta x_i} - \theta x_i + 1\right)}{(1+\beta e^{-\theta x_i})^4 h(x_i)^2} - \frac{x_i e^{-\theta x_i}}{1+\beta e^{-\theta x_i}} + \frac{e^{-2\theta x_i}\beta x_i}{(1+\beta e^{-\theta x_i})^2}\right).$$

$$I_{\lambda\alpha} = I_{\alpha\lambda}.$$

$$I_{\lambda\beta} = I_{\beta\lambda}.$$

$$I_{\lambda\lambda} = -\sum_{\imath=1}^{m}\left(-\frac{e^{\lambda x_\imath} x_\imath \alpha\left(\alpha(\lambda x_\imath - 2)e^{\lambda x_\imath} - \lambda x_\imath - 2\right)}{(1+\alpha e^{\lambda x_\imath})^3 h(x_\imath)} - \frac{\alpha^2 e^{2\lambda x_\imath}\left(\alpha e^{\lambda x_\imath} + \lambda x_\imath + 1\right)^2}{(1+\alpha e^{\lambda x_\imath})^4 h(x_\imath)^2} - \frac{\alpha x_\imath^2 e^{\lambda x_\imath}}{1+\alpha e^{\lambda x_\imath}} + \frac{\alpha^2 x_\imath^2 e^{2\lambda x_\imath}}{(1+\alpha e^{\lambda x_\imath})^2}\right).$$

$$I_{\lambda\theta} = -\alpha\beta\sum_{\imath=1}^{m}\left(\frac{\left(\alpha e^{\lambda x_\imath} + \lambda x_\imath + 1\right)e^{x_\imath(\lambda-\theta)}\left(\beta e^{-\theta x_\imath} - \theta x_\imath + 1\right)}{(1+\alpha e^{\lambda x_\imath})^2(1+\beta e^{-\theta x_\imath})^2 h(x_\imath)^2}\right).$$

$$I_{\theta\alpha} = I_{\alpha\theta}.$$

$$I_{\theta\beta} = I_{\beta\theta}.$$

$$I_{\theta\lambda} = I_{\lambda\theta}.$$

$$I_{\theta\theta} = \sum_{\imath=1}^{m}\left(-\frac{\left(\beta(\theta x_\imath + 2)e^{-\theta x_\imath} - \theta x_\imath + 2\right)e^{-\theta x_\imath} x_\imath \beta}{(1+\beta e^{-\theta x_\imath})^3 h(x_\imath)} - \frac{\beta^2 e^{-2\theta x_\imath}\left(\beta e^{-\theta x_\imath} - \theta x_\imath + 1\right)^2}{(1+\beta e^{-\theta x_\imath})^4 h(x_\imath)^2} + \frac{\beta x_\imath^2 e^{-\theta x_\imath}}{1+\beta e^{-\theta x_\imath}} - \frac{\beta^2 x_\imath^2 e^{-2\theta x_\imath}}{(1+\beta e^{-\theta x_\imath})^2}\right).$$