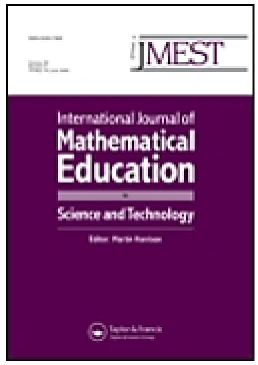

Transitional points in constructing the preimage concept in linear algebra

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Transitional points in constructing the preimage concept in linear algebra

From an APOS (Action—Process—Object—Schema) theory perspective, learning mathematics involves construction of knowledge through mental mechanisms, which evolves between different mental structures or stages. The focus of this study is to explore how transition occurs from an Action to a Process conception, in the context of a task related to the learning of the concept of linear transformation in general, and the notion of preimage in particular. A questionnaire was designed and applied to 31 students from three different higher education institutions in Mexico, who were enrolled in an introductory linear algebra course. For the first time, transitional points known as *levels* are explicitly identified in the aforementioned context and empirical evidence is presented. Some difficulties resulting from the intervention of constructions related to other concepts are also pointed out. Discussion about the characteristics of levels in the APOS framework provides a theoretical contribution to the field.

Keywords: mental structures; mental mechanisms; levels in APOS theory; linear transformation; preimage; linear algebra

Introduction

The process of learning in general, and of mathematics in particular, can ideally be thought of as progressing in terms of knowledge, abilities, insights and expertise within cognitive, social and affective realms. This advancement involves, again ideally, passages that allow an individual to deepen this experience, opening new perspectives and enabling different interpretations. Transitions from one state to another are necessary parts of this journey.

In the field of Mathematics Education, issues related to transitions have been addressed by some researchers from different angles. Yerushalmy (2005) describes the notion of *critical transition* as “a learning situation that is found to involve a noticeable change of point of view” (p. 37) and notes that it is accompanied by a change of lenses through which the concept is seen. Gueudet et al. (2016) identify two types of transitions: the ones that involve conceptual changes and those that occur between different social contexts. These authors pose an interesting question about the nature of transitions in mathematics learning. Is the learning process continuous or are there ruptures? In their literature review, they found studies that offer results for both viewpoints.

APOS (Action—Process—Object—Schema) theory, a framework about knowledge construction, does not concern itself with the philosophical question of continuity/discontinuity, however transitional points in learning are at the core of its explanation about how knowledge is constructed. According to this theory, an individual progresses from one stage of knowledge construction to another by means of mental mechanisms. For example, an Action stage, characterized by being dominated by external cues and algorithms, gets transformed into a Process stage after the individual repeats and reflects on those Actions and interiorizes them. Studies performed from an APOS perspective generally focus on the mental structures, that is the stages of Action, Process and Object, and less is known about the mental mechanisms that give rise to the transitions and how they take place.

A not-so-well-known aspect of APOS Theory is the existence of *levels*—transitional moments that can be observed between stages, in line with the work of Piaget (1974/1976; 1975) and Dubinsky et al. (2013). More information will be given about the nature of these *developmental junctures* (Arnon et al., 2014, p. 139) later in the article; here we mention as an example that an individual with an Action conception might pass through some levels *before*, or better said, *on the way* to constructing a Process conception.

In this paper we focus on the levels between the stages of Action and Process in the context of a task that is designed specifically to identify the nature of these transitions related to the learning of the linear transformation concept in general, and the notion of preimage in particular.

A major contribution of this study is the discussion about characteristics of levels as learning junctures between two stages, when progressing from an Action to a Process conception. Another aspect consists in providing evidence for the different transitional points between the mentioned structures, identifying associated difficulties as well as issues linked to connections of linear transformations with other concepts such as basis, matrices and dimension.

In the rest of this paper we first present a literature review as related to the functional aspects of linear transformations. Next we describe briefly the APOS Theory, and the notions of *stage* and *level*. We then present the task design as well as the related data, including difficulties observed, discussing the results obtained. Afterwards we point out some open questions and offer suggestions for future research. Some pedagogical strategies are mentioned in the light of our research findings and finally a Discussion section concludes by overviewing the contributions of this study.

Literature review on linear transformations related to functional aspects

Although the linear transformation concept has been investigated from different angles by different researchers (Andrews-Larson et al., 2017; Figueroa et al., 2018; Oktaç, 2018; Sierpinska, 2000), its functional aspects have been the subject of considerably fewer number of studies. Zandieh et al. (2017) explore the relationship between a high school function conception and a university linear transformation conception. These authors advocate a unified function-transformation concept; according to APOS theory this happens when the vector space Object gets assimilated into the function Schema as a possible domain (Roa-Fuentes & Oktaç, 2010). The relationship between these two concepts was also studied by Bagley et al. (2015) in connection with the concepts of inverse, composition and identity. Our literature search did not reveal any works on the concept of preimage in relation with linear transformations, although it has been studied in a more general functional context (Markovits et al., 1986) and that of geometric transformations (Hollebrands, 2003). In Villabona et al. (2020) a study is presented in which a student's construction of the concepts domain, image and preimage of a linear transformation was explored.

Theoretical considerations

In this study APOS theory is adopted as a framework, as it provides a cognitive approach applied in the context of the understanding of advanced mathematical topics. The basic elements of this theory are known as *mental structures*, *stages* or *conceptions*, that are constructed by means of mental mechanisms. *Actions* are driven externally, where the individual can transform previously constructed Objects. *Processes* are developed when an internal stimulus replaces the external algorithms or rules via the mechanism *interiorization*. When Processes are *encapsulated* they become *Objects* to which Actions or Processes related to other concepts can be applied. Finally all these structures and their relationships can come together to form part of a *Schema*. In this study we focus on Actions and Processes as building blocks of mathematical learning. Following the common convention, the names of the stages will begin with a capital letter, to distinguish them from daily usage of the same words.

There are two other mechanisms in APOS theory that are worth mentioning. *Coordination* is the mechanism through which two or more processes are united to give rise to a new Process, which carries the properties of each component, but contains new information. Through *reversal* a Process can be reverted, so that the original input elements that gave rise to the output elements can be obtained starting with these latter ones. The construction of the preimage concept can be thought of as related to the reversal of an Image Process. These mechanisms will be clearer when examples are provided in the data analysis section of this article.

Whereas all studies from an APOS viewpoint work with mental structures or stages, only two research studies so far touched on levels between stages in their data analyses: Dubinsky et al. (2013) on infinite processes and Arnon (1998) on study of fractions at the elementary school. The following are the main characteristics of stages and levels, in relation with each other:

> **A stage cannot be skipped.** If it is, the subject's understanding of the concept will lack coherence. Thus, stages are sequential, with each stage necessary for development of successive stages.
>
> **A level may or may not be reflected in the data of a specific subject.** This is because the subject may be able to move to the next level or stage rapidly so that the level is skipped, done very quickly, or is not observable in the already acquired higher level or stage.
>
> **Stages are invariant over topics and are part of the general theory.** Levels will be different for different concepts (Dubinsky et al. 2013). In many works, Piaget gave examples in which the development of different concepts gave rise to different levels. (Arnon et al., 2014, p. 139)

We add to this description that levels evidenced in different individuals might represent different paths from one conception to another. We also underline that a level, although not as apparent as in a stage, represents a major shift in thinking about a concept.

Task design and method

A task was designed from an APOS perspective, as part of a questionnaire applied to a total of 31 students from three different higher education institutions in Mexico, who were enrolled in an introductory linear algebra course. In its design algebraic, geometrical as well as functional aspects of a linear transformation were used, integrating in this way different facets of this concept. By a functional aspect we mean considering the linear transformation as a correspondence that associates the vectors of a domain space to the vectors of an image space. The task in question was presented as follows (translated from Spanish):

Consider the linear transformation $T{:}R^2 \to R^2$ associated to the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

a) Determine its domain.

b) Determine its image.

c) Does the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ belong to the image of the transformation? If the answer is YES find the preimage and graph it. If the answer is NO, justify.

d) Graph the domain of T.

e) Graph the image of T.

f) Does the vector $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ belong to the image of the transformation? If the answer is YES find the preimage and graph it. If the answer is NO, justify.

As part of the task design, the linear transformation was chosen to be non-injective, so that a given vector in the image space would correspond to a set of infinitely many vectors in the domain space.

Since in this article we are mainly interested in the notion of preimage, we present an initial theoretical analysis only for that notion. An Action conception would involve working with specific vectors, as in finding a vector in the domain as a preimage, given an image vector. Constructing a Process conception of preimage requires the individual to identify the whole set of vectors in the domain that are related to a specific vector in the image through the linear transformation. It also requires determining the characteristics that vectors in the domain should satisfy in order to form part of the preimage set. We note that an individual might construct either an algebraic or a geometric Process conception without necessarily having constructed the other. Note that levels do not form part of these initial theoretical considerations, since they can differ from person to person, or be skipped altogether.

For the analysis of data we used the specific methodology associated to APOS Theory (Arnon et al., 2014) where each researcher examines the students' answers and writes down the important points to consider, focusing on the mental constructions and their evidence. Afterwards these individual analyses are compared and discussed until an agreement is reached.

Now we turn to our data and present the levels that we discovered between the stages of Action and Process in relation with the concept of preimage, in the context of a linear transformation. Before we start with the description of the Action stage, we present the case of a student as an example for not having constructed an Action conception. We end with the description of the Process stage; in between are the four levels identified as transitional points.

Data and their interpretation

In this analysis we center our attention in students' answers to part (f) of the task, but in some cases we present evidence from the other parts of the question as necessary. As described earlier in this paper, an Action conception is externally driven by means of algorithms, formulas or a trial-and-error approach and implies finding a specific vector from the domain set, as a preimage of a given vector from the image set. A Process conception involves internal control over the concept and it entails imagining the whole set of preimage vectors for a given vector in the image space. In what follows students are named as E, followed by a number to distinguish between them.

Oftentimes an Action conception is considered as having difficulties or not being able to answer a question. In APOS Theory Actions play an important role as building blocks of knowledge construction. To illustrate the difference between these two situations, we first present the case of a student who has not constructed an Action conception.

Before an Action conception

In order to differentiate between an Action conception and the inability to solve a problem (sometimes referred to as pre-Action), we start by presenting E1's work. This student calculates the range of the given matrix as 1 and draws the vector (1, 1) as the image of the given transformation. For part (f) he first indicates that the vector (5, 5) "belongs since it is a multiple of the image" (referring to the vector (1, 1)). Then he writes a formula for calculating the inverse of a matrix and concludes that the inverse "cannot be obtained" as shown in Figure 1.

Figure 1 near here

Hence this student conditions the possibility of finding the preimage set to the existence of the inverse of the given matrix, that is, to the injectivity of the transformation. E1 has not constructed an Action conception about the preimage concept yet, since he cannot find any vector that belongs to the preimage set.

As mentioned before, once a Process is constructed, the mechanism of reversal allows the individual to go back to the original setting, starting with the output of the Process. Sometimes the reverse path can correspond exactly to the inverse Process, such as finding the inverse of an invertible function. Other times however, an exact inverse path may not exist (Fabián Campos, 2017), such as in the case of our example, a non-invertible linear transformation. In these cases it does not mean that the mechanism of reversal cannot be employed; it rather implies that the reverse path may be harder to construct, and the result may correspond to a set of possible solutions. In the case of a non-invertible linear transformation, the image of a domain vector can be found by directly applying the transformation or its matrix to the given vector; it is obtained in a straightforward manner. However the return path is not exactly the original path walked backwards.

Action conception

E2 identifies the image of the given transformation as $Img(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in R^2 | \; x = y \right\}$ and draws its graph correctly. She responds part (f) indicating correctly that the vector $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ belongs to the image of the given linear transformation and showing that the image of $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ under that transformation; she does this by multiplying the given matrix by the vector she has chosen. However there is no indication in her production that implies the consideration of a set of values; she rather finds a particular vector as the preimage and draws both that vector and the given image on the graph as line segments with arrows, as shown in Figure 2.

Figure 2 near here

Towards a Process conception

When a student can consider more than one vector as part of the pre-image of a given vector, a progress in the construction of this concept from an Action to a Process conception is implied, since the no-injectivity of the linear transformation is taken into account, although it may not be declared explicitly. Some interesting questions can be posed about this situation. What kind of a reasoning does the student employ in order to find such vectors? It may not be easy to answer this question, but responses to the following ones may offer an insight. What kind of relationships does the individual establish between those vectors? How much of the property of constituting the preimage of a given vector under a given transformation is present in that relationship? How is the student thinking about these vectors in connection with a set?

When a student displays all the elements of an Action conception and some of the elements of a Process conception without having constructed the latter fully, we may be witnessing a level between these two stages. In what follows we describe four stages as evidenced by the data, explaining the kind of shift that occurs in each one. We remind the reader that the names of these levels reflect their content specificity and they cannot be generalized to other concepts.

Level "preimage as two vectors"

Before starting to work on different parts of the problem, E3 identifies the columns of the matrix as images of the canonical basis vectors as shown in Figure 3, determining in this way the given transformation in terms of the basis:

Figure 3 near here

E3 identifies the image of the transformation as "Im = < (1, 1) >" and graphs it correctly.

The focus on basis vectors is carried to part (f) as well; E3 writes that the given vector belongs to the image and working with the canonical basis identifies (5, 0) and (0, 5) as preimage vectors as shown in Figure 3. He also graphs these two vectors as line segments with arrows.

Figure 4 near here

E3 looks for vectors whose elements sum up to 5; multiplication of the canonical basis vectors provides them. His conception of basis on the one hand helps him to relate the concepts in question. On the other hand it limits his search to the multiples of the canonical basis vectors. He might be thinking that as in the case of the image set of the given linear transformation, its preimage space can somehow be determined in terms of two vectors. It also seems that this student is not thinking in terms of expressing the vectors of a certain set as linear combinations of basis vectors. Actually he only takes into account the multiplication by a scalar of the basis vectors, the addition component being missing. Possibly he does not coordinate the Processes related to the addition and the multiplication by a scalar of vectors, from which the Process of linear combination would be constructed.

The two vectors that he proposes as the preimage are related to each other by being multiples of the canonical basis vectors, that is they are obtained by multiplying each canonical basis vector by 5. The progress obtained compared to an Action conception is accepting that the preimage does not have to consist of a unique vector. We also note that these two vectors are not expressed as forming a set. In fact, the student writes as the solution: "T(5,0) = (5,5) or T(0,5) = (5,5)". It is as if E3 is shy about admitting that two vectors at the same time can belong to the preimage.

Level "preimage as a set of two vectors"

E4 writes that $Im(T) = \ <\begin{pmatrix}1\\1\end{pmatrix}>$ and graphs the line correctly. She also mentions that "T is not injective hence it does not have an inverse". For part (f) she indicates that the given vector belongs to the preimage and offers a two-vector set as a solution, as shown in Figure 5:

Figure 5 near here

Apart from considering more than one vector as part of the preimage (probably due to the no-injectivity of the transformation), E4 clearly expresses the solution as a set. It is noteworthy that the two vectors are related to each other by having the same elements but in different order. Here we are starting to see the preimage as a set, whose properties are not yet clear and hence it cannot be determined completely; however this is a big step towards a Process conception. This student does not offer a graph as part of the solution.

Level "preimage as multiple solutions"

E5 expresses the image of the transformation as "$k. < 1, 1 >,\ k\, \epsilon \mathrm{R}$" and draws the geometric representation of the line correctly. For part (f) he indicates that the given vector belongs to the image "because x=y". He then indicates that the system given by the matrix equation shown in Figure 6 "has multiple solutions in particular (2, 3)", but does not graph the solution.

Figure 6 near here

Although it is not clear whether E5 is thinking about a finite or an infinite number of vectors, the fact that he starts seeing the preimage as a solution set associated to a matrix equation that he set up is clearly a progress from an Action conception. However this set is not represented explicitly, neither in an algebraic nor in a graphical form.

Level "preimage as infinitely many preimages"

E6, after correctly graphing the image set of the transformation as a line, in part (f) mentions that the given vector belongs to the image and writes the following: "determinant of the matrix $= \frac{1}{1-1} = \frac{1}{0}$! The determinant is not defined, therefore there is no inverse matrix, however the preimage can be the $\binom{4}{1}$, the $\binom{1}{4}$, the $\binom{5/2}{5/2}$, that is there are infinitely many preimages." This is shown in Figure 7.

Figure 7 near here

It is interesting to note that E6, as all the other students who gave more than one vector as part of the preimage, identifies a pair of vectors as having the same elements but in different order. He also gives a third example as having equal elements, satisfying the property that the sum should be 5. However the solution is not expressed as a set; actually this student might be thinking about each of these vectors as a different preimage. The progress consists in considering infinitely many solutions as possible preimage vectors. Nevertheless the preimage does not appear explicitly as a set that satisfies a certain property, nor does the student graphs the solution.

Process conception

A Process conception is achieved when an individual repeats the Actions, reflects on them and interiorizes them. This way they can control the Actions, think about the properties of the concept and work with it in a general manner as opposed to dealing with specific instances. They can imagine performing the Actions in their minds, and not necessarily by following an explicit rule each time. As mentioned earlier, a Process conception can be constructed related to different representations of a concept, for example a Process conception of an algebraic character, or of a geometric one. When different processes associated to different representations are coordinated, we can talk about a Process conception of the related concept.

Algebraic Process

Student E7 expresses the image set as "the line generated by $\{\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} t \\ t \end{pmatrix} : c \in R\}$".

Although his notation is somewhat sloppy, the fact that he draws the line correctly shows that he is thinking about the line made up of those points whose coordinates are equal to each other.

For part (f), E7 indicates that the given vector belongs to the image, writes that "they are all the vectors such that x+y=5   x= 5-y" and then gives the preimage set as the parametric equation of the corresponding line, as shown in Figure 8.

Figure 8 near here

We can see in this work the general expression of the set of vectors and the formulation of the property that they satisfy, that are indications of a Process conception. This student did not graph his solution.

Coordinated Process

In order to find the image of the linear transformation, E8 applies the given matrix to the canonical basis vectors, obtaining $\begin{pmatrix}1\\1\end{pmatrix}$ each time. She then concludes that "$\{\begin{pmatrix}1\\1\end{pmatrix}\}$ is basis of the image" and expresses the image set as "$\{\begin{pmatrix}x\\y\end{pmatrix} \in R | x = y\}$". She also graphs the image correctly. In part (f) she mentions that the given vector belongs to the preimage and represents it both in a set notation and graphically (Figure 9).

Figure 9 near here

The generality of her answer, the use of the mathematical notation, the connection between the two representations evidence a Process conception of preimage. Here there is no need to mention specific vectors, since the answer is given in terms of satisfying a property and compacted into a set notation.

Looking back at the data

Now that we have shown examples for each stage of Action and Process, as well as the levels found between these two conceptions for the preimage concept, we think it would be useful to return to some of the data and the information that they convey.

An Action conception of preimage is characterized by finding a specific vector in the domain whose image is the given vector. When the student starts moving towards a Process conception, they may pass through one or more levels. The levels are identified as having all the elements of an Action conception, and some of the elements of a Process conception but not having developed the latter completely. Each level has its own characteristics and is named alluding to its particularities; levels do not have to be sequential.

Since levels are concept specific and depend on the participants' characteristics, each individual may follow a different path from an Action to a Process conception; some of these paths may join each other or have common crossings (see Figure 16 in the Discussion section).

In order to explain better the difference between an Action and a Process conception and the role of levels as transitional points, let's analyze with some more depth E6's response. This student evidences that he has constructed an Action conception, since he can find specific vectors whose image is the given vector. We can also see a progress towards Process while he mentions a first, a second, and a third vector—hence accepting the existence of more than one vector as preimage— to conclude at the end that there are infinitely many preimages. His conception is quite close to a Process, but it is not general enough and does not provide the properties of the set that is being asked for. So the Action of finding specific vectors as preimage vectors are not completely interiorized in order to open the way to imagining and expressing the preimage set as a collection of vectors that satisfy a certain property.

Difficulties observed in students' work

Many published articles in linear algebra education start by mentioning that this is a difficult subject for students to learn and for instructors to teach. In this section we offer evidence based on data, of the kinds of difficulties related to the function concept and the concept of dimension, in the construction of the preimage concept, from the viewpoint of APOS theory.

Difficulties related to the use of the function concept

Some students had difficulties in identifying the domain and/or the image of the linear transformation, because of a strong connection that they established with the function concept, as learned in calculus courses.

E9 graphs the domain and image of the given transformation as shown in Figure 10.

Figure 10 near here

As domain, E9 signals the x-axis on an xy-plane and writes R as the answer. As image he signals the y-axis, omitting the origin, since he writes R \ {0} as the answer. Apparently this student is thinking about real-valued functions of a single variable, trying to apply that knowledge in the context of linear transformations. It is not clear why zero is left out; it may be because he remembers vaguely working with certain functions that are undefined for zero such as f(x)=1/x but confuses the domain and image sets. For part (f) he produces the answer shown in Figure 11:

Figure 11 near here

He writes: “Yes it belongs”; it is not clear why he draws the vector (5, 5). We don’t know if he does it to have produced some graph because he doesn’t have an answer for the preimage part of the problem, or if he considers this vector as a preimage of itself.

Another student, E10, also considers the set of real numbers, as shown in figure 12.

Figure 12 near here

For parts (d) and (e), this student writes respectively: “The domain is in the reals” and “The image is in the reals”. For part (f) he denominates (5, 5) as the “vector”, (10, 10) as the “image” and (3, 2) as the “preimage”. So, it looks like he finds an image and a preimage for the given vector (5, 5). The reason behind choosing (10, 10) as the image vector is not clear. There is a lack of connection between the vectors he draws and the domain and image sets that he represents as a sort of Venn diagram, probably influenced by a set-theoretic definition of functions.

Both students draw from their knowledge of real variable functions from Calculus as they try to answer the given questions. Perhaps the only time that they remember having seen the concepts of domain and image was in the context of functions of real variables and they try to apply this knowledge to linear transformations. In order to construct the linear transformation concept starting with the notion of function, the student needs to view vector spaces as possible domain and image sets (Roa-Fuentes & Oktaç, 2010). Neither E9 nor E10 seems to be aware that the given transformation is defined on R^2, although the question explicitly states it. Relying on the function concept without its contextualization to linear algebra can become an obstacle in understanding the notion of transformation.

Difficulties related to the dimension concept

We present the case of E11, who indicates a particular vector as preimage, but her reasons are related to her conception about the dimension of a vector space. E11 indicates that the dimension of the domain is 1, and the image and domain sets are equal to each other, characterized as "$\begin{pmatrix} x \\ x \end{pmatrix}$ $x \in R$", as shown in Figure 13.

Figure 13 near here

She graphs the image correctly, but indicates that it is also the domain, as shown in Figure 14.

Figure 14 near here

E11 then proceeds to part (f) and says that (5, 5) belongs to the image and that the preimage is $\begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$, as shown in Figure 15.

Figure 15 near here

Since E11 had determined the domain set as a one-dimensional subset containing of those vectors whose coordinates are equal to each other, the only vector that satisfies that condition and at the same time whose coordinates add up to 5, is (2.5, 2.5). We can see how, the use of the dimension concept, which has not been constructed as a Process, interferes with the preimage concept. The graph that the student drew shows a line generated by the vector (1, 1) (see Figure 13).

Continuation of research

The present study is a first look at the levels that might be present during the learning of the preimage concept in the context of linear transformations. It provides a first insight into the nature of these transitional points; however more research is necessary in order to deepen the discussion about these junctures, what motivates them and the role they play in the construction of mathematical concepts. In particular we suggest designing a study in which the research instruments are revised to focus on the related notions and interviews are performed to observe the students while they work on problems. That way the information given in this article can be consolidated and new findings taken into account for the understanding of the preimage of a linear transformation, a topic much underrepresented in current investigations.

In the new task design the following elements can be taken into account: change the vector in part (c) so that its image would not coincide with the vector given in part (f); ask the student to define ‘domain’, ‘image’ and ‘pre-image’; explore different representations and their connections; include other questions that would help in compiling information about the phenomena observed in this study.

One direction that a future study can take is the identification of levels between Process and Object conceptions, for the same concept studied in this research. This would imply the design of a suitable research instrument, and the selection of an appropriate population of students. The mathematical problem would be designed so that a student with a Process conception can start working on it, but without an Object conception cannot reach a whole solution. It would be interesting to compare its results with those obtained in this study.

Representations and different modes of thinking have an intimate connection with the construction of mathematical concepts. A synthetic-geometric representation is perceived directly by an individual (Sierpinska, 2000); for example a line drawn on a piece of paper probably evokes more or less the same image in different individuals who see it. On the other hand, if a concept is represented in an analytic mode, such as in the case of a system of equations (Sierpinska, 2000), the symbols tell a whole different story to the person who has studied those mathematical objects than to a person who has no knowledge of math. A future study can address different representations related to the notions of domain, image and preimage in the context of linear transformations, with the goal of examining how their coordination takes place and how the related Process is constructed.

The mechanism of reversal merits a study of its own, perhaps in the context of functions, to develop an understanding of its working. The relationship between the construction of a Process, its input and output elements and the way these are connected in students' minds make up an interesting topic to study, which would have implications in several areas of mathematics. Similarly the mechanism of coordination can be investigated, focusing on how it takes place; Arnon et al. (2014) have a proposition about the way Processes and Objects associated to two concepts can be joined to give rise to a new Process, but as of yet no empirical study has taken it up.

Other ideas for future research include: selection of different domain and co-domain spaces; comparison of the reversal mechanism when preimage set and inverse transformation are explored; consideration of the relationship between image and preimage sets and related processes; study of the mechanism of interiorization in view of levels.

Discussion

In this article, for the first time explicit levels between two conceptions, namely Action and Process are presented from an APOS viewpoint, together with their characteristics, based on evidence from data collected from linear algebra students in three different institutions. This is done in connection with the concept of preimage of a linear transformation, a topic on which we did not find a direct study.

Earlier studies mentioned certain characteristics of levels in APOS Theory (see Arnon et al., 2014). In this paper we add two new properties to that description: that levels evidenced in different individuals might represent different paths from one conception to another, and that a level, as a transitional point in learning a concept, represents a major shift in thinking about it. These aspects have been illustrated with empirical evidence in this paper. We hope that further research can look into different mathematical topics in order to identify the levels that may be involved in transiting from a conception to another. Figure 16 suggests a way to visualize the levels in relation with stages.

Figure 16 near here

In this paper some difficulties experienced by students in connection with the preimage concept where other notions of linear algebra such basis, dimension, rank, matrix are discussed. One aspect that stands out in the difficulties observed is the lack of coordination between different Processes; this would be an area in which didactical strategies can be centered. Another area would be establishing a continuation between functions as studied in Calculus courses and the linear transformation concept as seen in a linear algebra course. Another pedagogical suggestion is to work with different kinds of functions in Calculus and linear transformations in linear algebra, not only injective ones.

Another difficulty corresponds to the graphical representation of vectors. Some students chose to use points and some others worked with directed line segments. The latter ones in general had difficulty in finding the requested preimage set, probably since perceiving the end points as forming a line is not as direct as in the case of using points.

We would like to underline the importance of task design in distinguishing transitional points in knowledge construction. In the case of levels between two stages in APOS theory, the task should be accessible to all students who have constructed a conception corresponding to one of the stages; it should also allow the manifestation of different kinds of progress between stages. Finally, the complete solution should be accessible to individuals with a conception related to the second stage in question. This way characteristics of different levels can also be identified.

Conflict of interest statement: We report that we do not have any potential competing interests.

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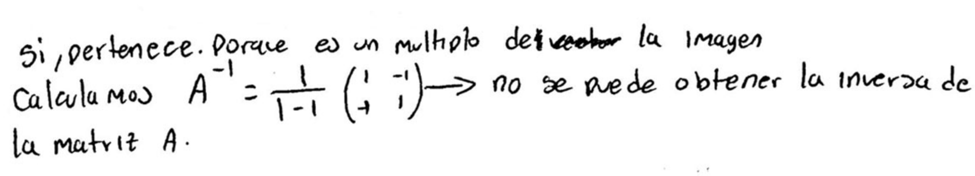
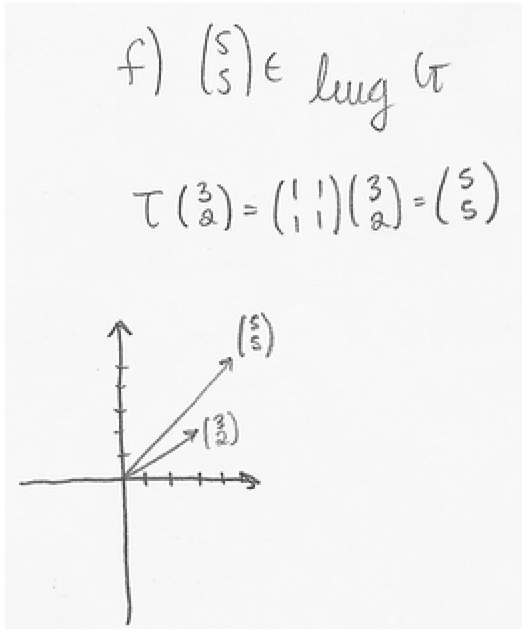


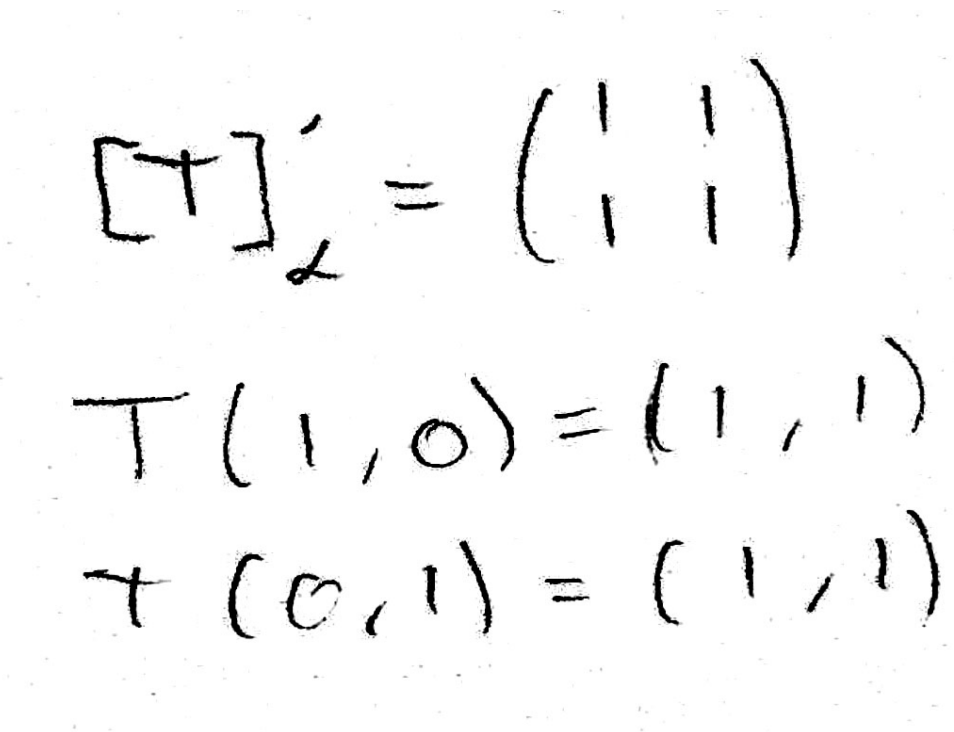
Preimage cannot be found since the matrix has no inverse

423x85mm (72 x 72 DPI)



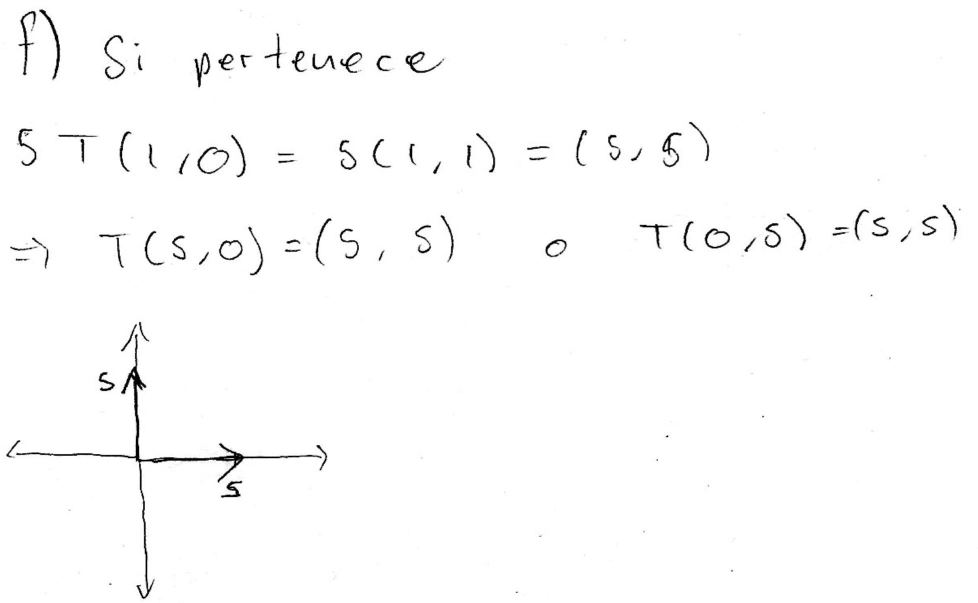
E1's solution to part (f)

22x26mm (300 x 300 DPI)



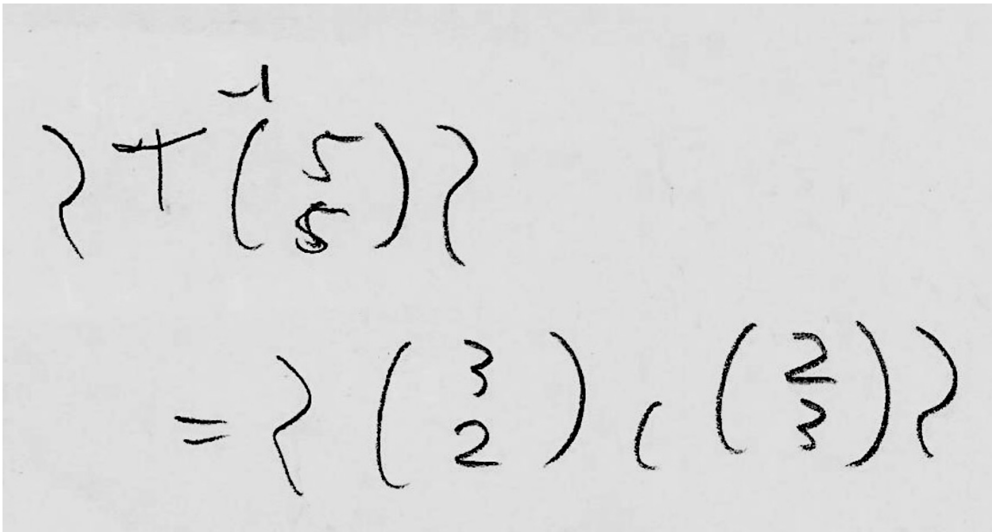
E3 expresses the matrix in terms of the basis vectors

374x271mm (72 x 72 DPI)



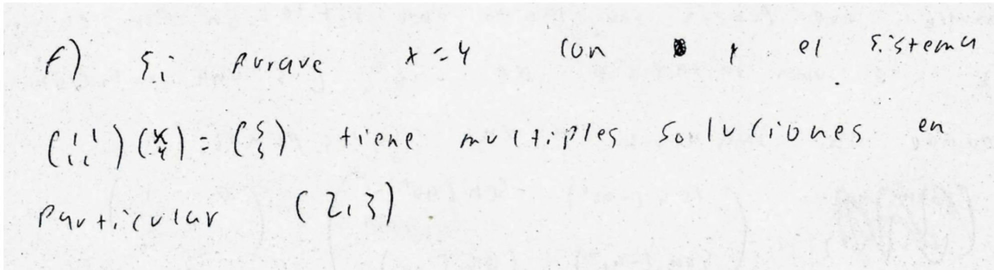
Preimage vectors according to E3

435x273mm (72 x 72 DPI)



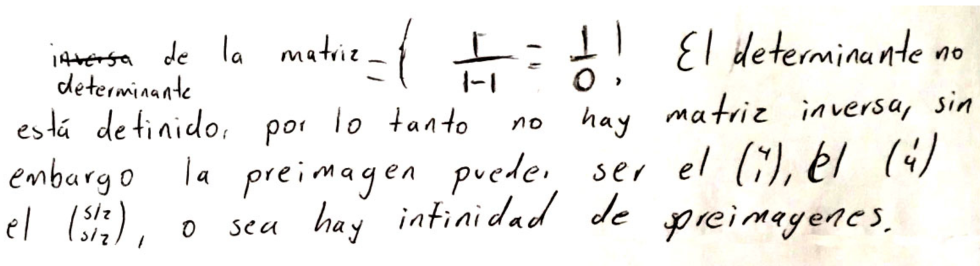
A set of two vectors as preimage

409x218mm (72 x 72 DPI)



Preimage as multiple solutions to a matrix equation

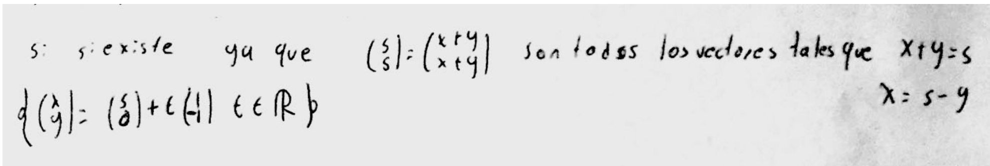
492x132mm (100 x 100 DPI)



~~inversa~~ determinante de la matriz $= \frac{1}{1-1} = \frac{1}{0}$! El determinante no está definido, por lo tanto no hay matriz inversa, sin embargo la preimagen puede ser el $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, el $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ el $\begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}$, o sea hay infinidad de preimagenes.

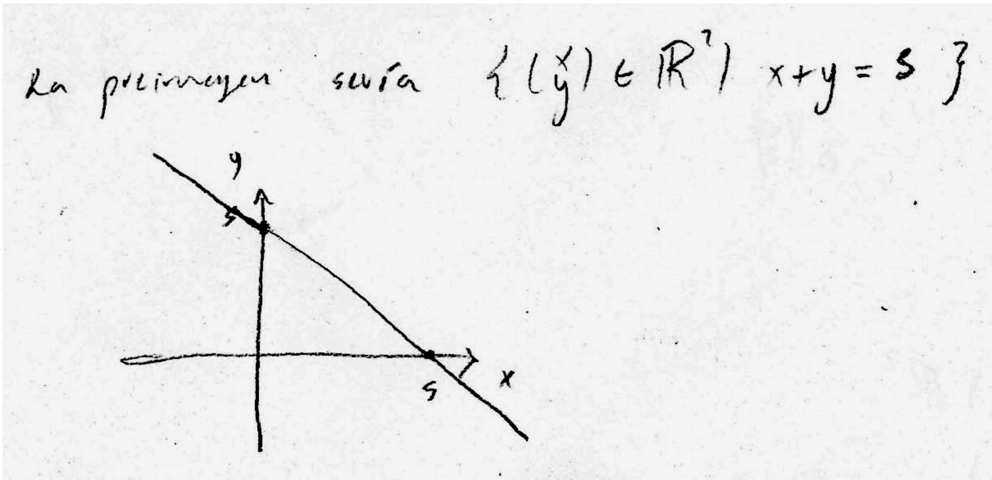
E6’s answer to part (f)

416x110mm (72 x 72 DPI)



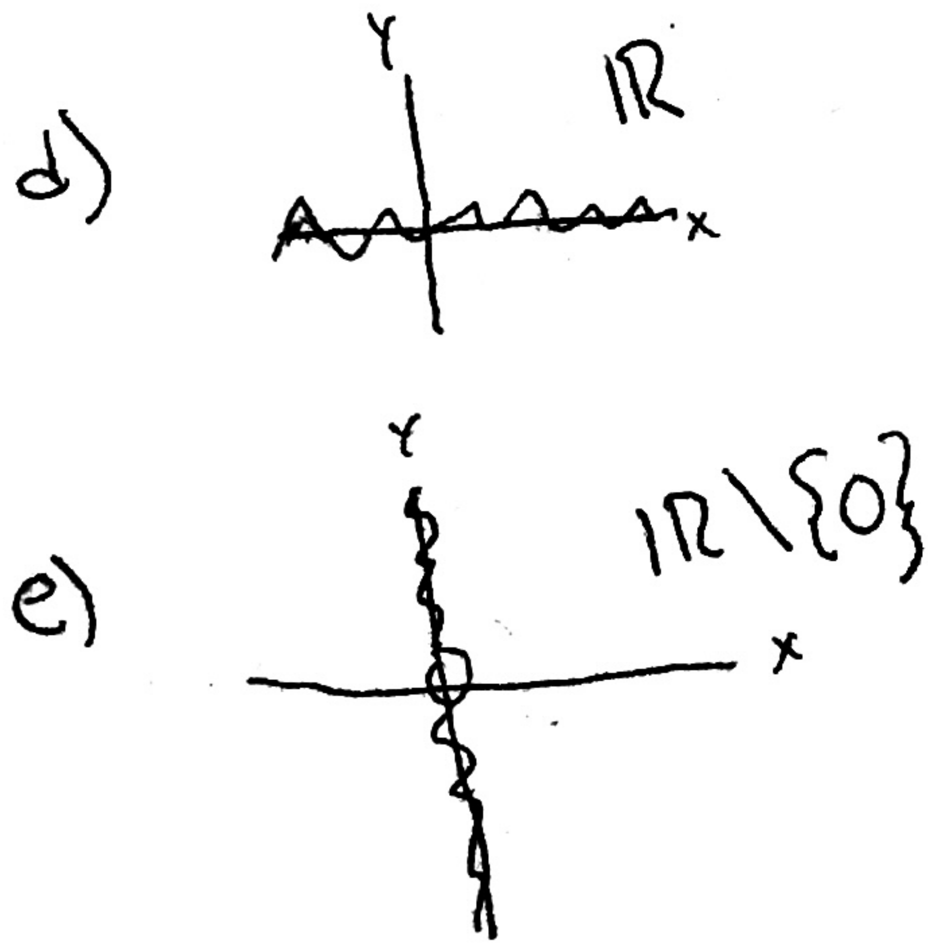
Preimage expressed as a set corresponding to a parametric line

488x81mm (72 x 72 DPI)



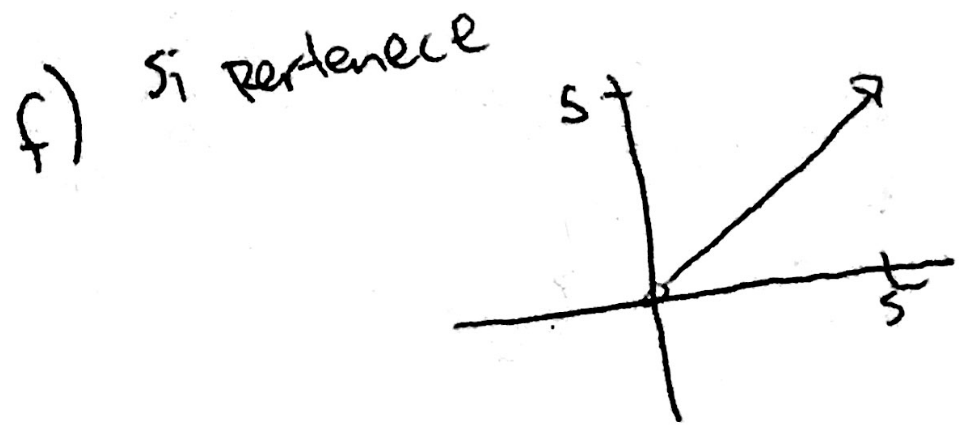
Preimage as a set and its graphical representation

430x208mm (72 x 72 DPI)



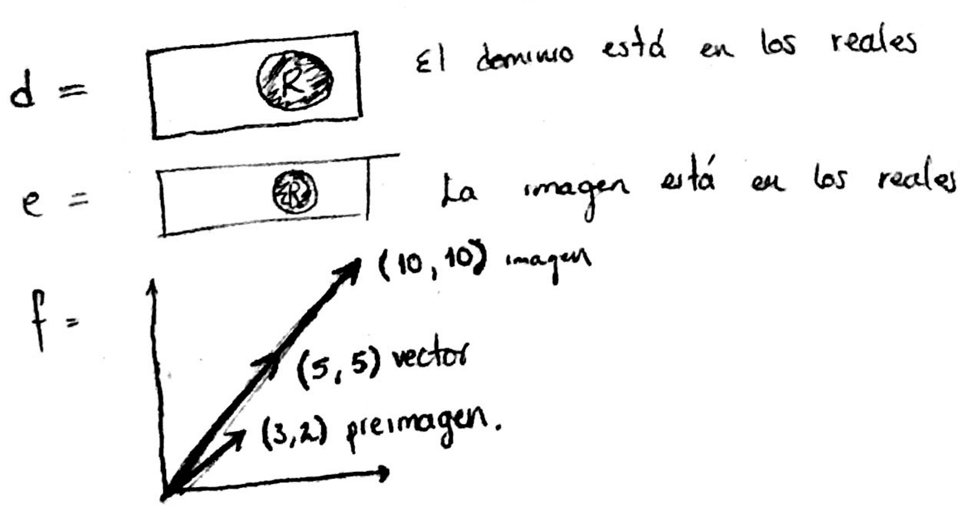
Domain and image according to E9

261x257mm (72 x 72 DPI)



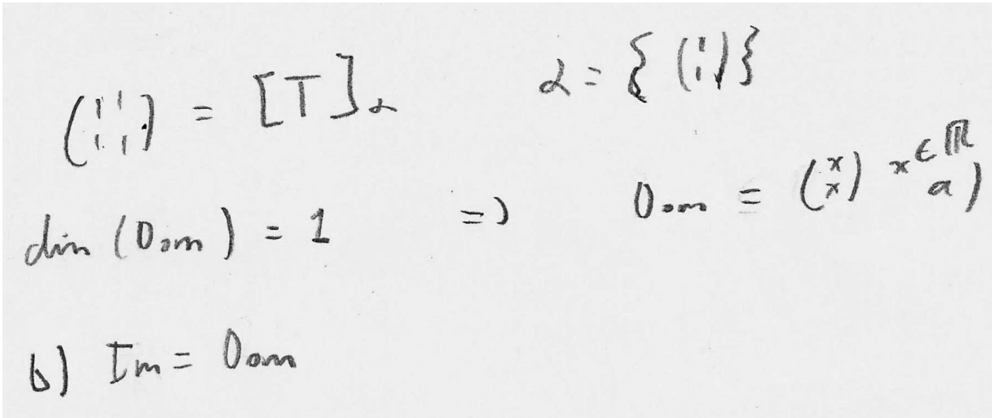
E9's work on part (f)

436x204mm (72 x 72 DPI)



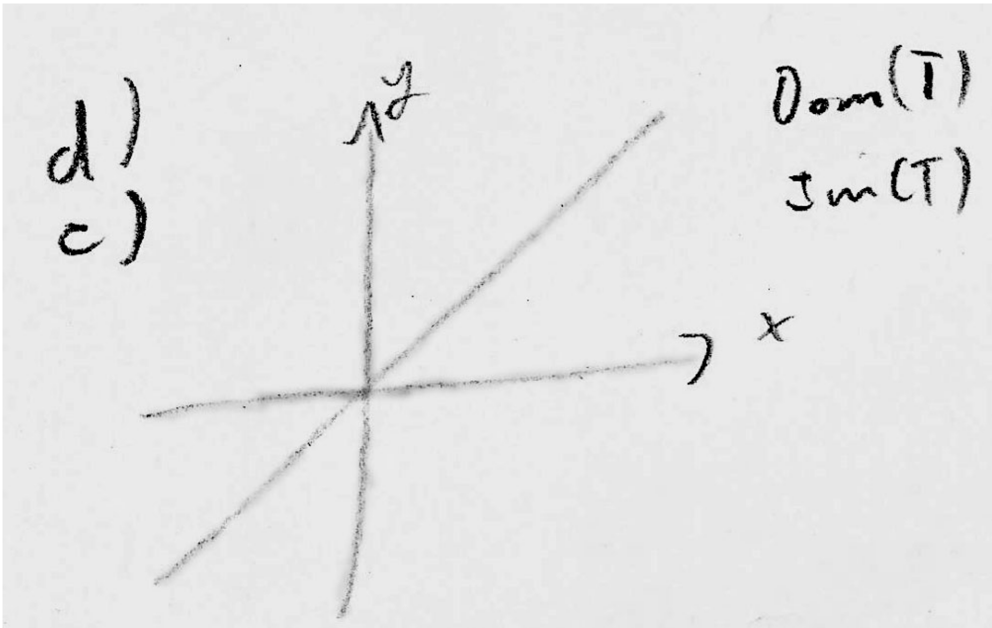
Domain and image according to E10

460x244mm (72 x 72 DPI)



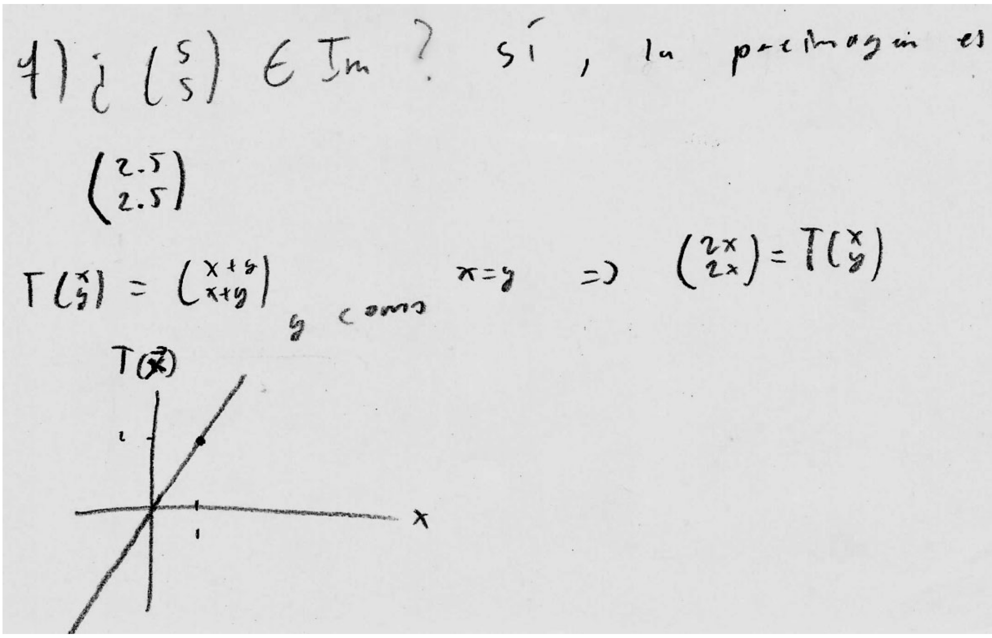
Domain and image according to E11

457x192mm (72 x 72 DPI)



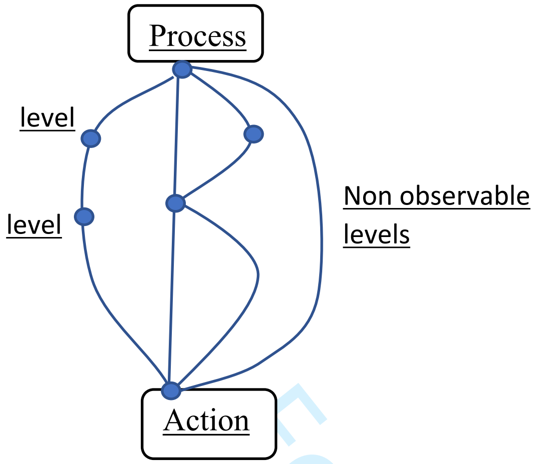
Graph of the domain and image according to E11

357x225mm (72 x 72 DPI)



Preimage as one vector in a one-dimensional domain

422x266mm (72 x 72 DPI)


Process
level
level
Non observable
levels
Action