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Soluciones exactas de EDPs lineales de segundo orden mediante álgebras

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TÍTULO DEL REPORTE TÉCNICO

Reporte final sobre proyecto: Soluciones exactas de EDPs lineales de segundo orden mediante álgebras

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TÍTULO DEL REPORTE TÉCNICO

Resumen del reporte técnico en español :

En el proyecto de investigación titulado

“Soluciones exactas de EDPs lineales mediante algebras”

se consideran ecuaciones diferenciales parciales (EDPs) lineales con coeficientes constantes de dos variables independientes de la forma

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0, \qquad (1)$$

y de tres variables independientes de la forma

$$Au_{xx}+Bu_{yy}+Du_{zz}+Eu_{xy}+Fu_{xz}+Gu_{yz}+ Gu_x+Hu_y+Ku_z+Lu=0. \qquad (2)$$

Inicialmente se propuso estudiar EDPs de segundo orden, pero el método propuesto se puede generalizar a orden superior; se consideran ecuaciones se tercer orden y cuarto orden para el caso de EDPs de dos variables independientes. Por ejemplo, se construyen familias de soluciones para las EDPs bi-armónica, bi-onda, y bi-telegráfica. La ecuación bi-armónica surge de la mecánica del medio continuo, incluida la teoría de la elasticidad lineal y la solución de los flujos de Stokes.

Para este trabajo es importante la “diferenciabilidad pretorcida”, tema que se desarrolló en un proyecto anterior. Para el caso de campos vectoriales se puede definir de la siguiente manera: dados dos campos vectoriales diferenciables φ, F, y un álgebra **A** de la misma dimensión que φ y F, se dice que F es φ**A**-diferenciable con derivada F'_φ, si F'_φ es un campo vectorial tal que la diferencial usual

$$dF_x= F'_\varphi(x)d\varphi_x, \text{ esto es, } dF_x(v)= F'_\varphi(x)d\varphi_x(v),$$

en donde se tiene $F'_\varphi(x)d\varphi_x(v)$ es el producto con respecto a **A** de $F'_\varphi(x)$ y $d\varphi_x(v)$ para todo vector v. Este tema se puede ver de manera más amplia en arXiv:1805.10524v13**.**

Con esta definición se tendrá que todas las funciones polinomiales en la variable w= φ(x) definidas con respecto a **A**, son φ**A**-diferenciables y se cumplen las reglas usuales de diferenciación, salvo la regla de la cadena. Lo mismo suceda para las funciones racionales, exponencial y trigonométricas de variable w= φ(x).

Como resultado de este proyecto se propone un método para construir familias de soluciones EDPs. Esta es parecida a la propuesta por P. W. Ketchum en 1928 y que se ha citado en trabajos recientes, por ejemplo, uno del matemático ucraniano Sergiy A. Plaksa de 2019. Ketchum considera una EDP y busca un álgebra tal que las funciones conjugadas (componentes) de funciones diferenciables en el sentido de Lorch, sean soluciones de la EDP. Para el caso particular de la ecuación de Laplace con dos variables dependientes, Ketchum encuentra que si **i** y **j** son los vectores unitarios canónicos del plano, se requiere que la suma de

los cuadrados de **i** y **j** sea cero, con respecto al álgebra que se considera la diferenciabilidad de Lorch. Nosotros proponemos considerar un campo vectorial planar afín φ y determinamos un álgebra A tal que la suma de los cuadrados de las componentes de φ sea cero. Bajo estas condiciones tendremos que las funciones conjugadas de las funciones φ**A**-diferenciables de segundo orden son funciones armónicas. Esta misma idea se puede aplicar para EDPs más generales que la ecuación de Laplace; se aplica para todas las EDPs de la forma

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0.$$

También se puede aplicar para ecuaciones de más variables dependientes como

$$Au_{xx}+Bu_{yy}+Cu_{zz}+Du_{xy}+Eu_{xz}+Fu_{yz}=0.$$

Más aún, dado φ se puede encontrar un álgebra tal que las funciones conjugadas de la función exponencial de w= φ(x) siempre van a ser soluciones de las EDP (1), y lo mismo para la EDP (2). Como ya se mencionó, el método se puede generalizar a ecuaciones de orden superior y de más variables independientes. El caso de solución de EDPs de dos variables independientes se puede ver en
https://authorea.com/users/405599/articles/526199-on-solutions-of-pdes-by-using-algebras.

Resumen del reporte técnico en inglés :

In the research project entitled

"Exact solutions of linear PDEs by means of algebras",

we consider linear partial differential equations (PDEs) with constant coefficients of two independent variables of the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0, \qquad (1)$$

and three independent variables of the form

$$Au_{xx}+Bu_{yy}+Du_{zz}+Eu_{xy}+Fu_{xz}+Gu_{yz}+ Gu_x+Hu_y+Ku_z+Lu=0. \qquad (2)$$

Initially, it was proposed to study second order PDEs, but the proposed method can be generalized to higher order; third order and fourth order equations are considered for the case of PDEs with two independent variables. For example, families of solutions for the bi-harmonic, bi-wave, and bi-telegraph PDEs, are constructed. The bi-harmonic equation arises from the continuum mechanics, including linear elasticity theory and the solution of Stokes flows.

Important for this work is the "pre-twisted differentiability", a topic that was developed in a previous project. For the case of vector fields it can be defined as follows: given two differentiable vector fields φ, F, and an algebra **A** of the same dimension as φ and F, F is said to be φ**A**-differentiable with derivative F'_φ, if F'_φ is a vector field such that the usual differential

$$dF_x= F'_\varphi(x)d\varphi_x, \text{ that is, } dF_x(v)= F'_\varphi(x)d\varphi_x(v),$$

where one has that $F'_\varphi(x)d\varphi_x(v)$ is the product with respect to **A** of $F'_\varphi(x)$ and $d\varphi_x(v)$ for every vector v. This topic can be seen more extensively in arXiv:1805.10524v13.

With this definition one will have that all polynomial functions in the variable w= φ(x) defined with respect to **A**, are φ**A**-differentiable and the usual differentiation rules are satisfied, except for the chain rule. The same is true for rational, exponential and trigonometric functions of variable w= φ(x).

As result of this project, a method for constructing families of solution PDEs, is proposed. This is similar to the one proposed by P. W. Ketchum in 1928 and cited in recent papers, for example, one by Ukrainian mathematician Sergiy A. Plaksa from 2019. Ketchum considers a PDE and searches for an algebra such that the conjugate functions (components) of differentiable functions in the Lorch sense, are solutions of the PDE. For the particular case of the Laplace equation with two dependent variables, Ketchum finds that if i and j are the canonical unit vectors of the plane, the sum of the squares of i and j is required to be zero, with respect to the algebra Lorch differentiability is considered. We propose to consider an affine planar vector field φ and determine an algebra **A** such that the sum of the squares of the components of φ is zero. Under these conditions, we will have that the conjugate functions of the second order φ**A**-differentiable functions are harmonic functions. The same idea can be applied for PDES more general than the Laplace equation; it applies for all PDEs of the form

$$Au\_xx+Bu\_xy+Cu\_yy=0.$$

It can also be applied for equations with more dependent variables, such as

$$Au\_xx+Bu\_yy+Cu\_zz+Du\_xy+Eu\_xz+Fu\_yz=0.$$

Moreover, given φ one can find an algebra such that the conjugate functions of the exponential function of w= φ(x) are always going to be solutions of PDEs (1), and the same for PDE (2).

As already mentioned, the method can be generalized to equations of higher order and of more independent variables. The case of solution of PDEs of two independent variables can be seen in https://authorea.com/users/405599/articles/526199-on-solutions-of-pdes-by-using-algebras.

Palabras clave: Función matriz exponencial, Ecuaciones diferenciales parciales (EDPs), Derivada de Lorch, Teoría de diferenciación.

Keywords: Matrix exponential function, Partial differential equations (PDEs), Lorch derivative, Differentiation theory.

Usuarios potenciales:

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Project report: Exact solutions of second-order linear PDEs by using algebras

1 Introduction

In this project PDEs of the forms

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0, \tag{1}$$

$$Au_{xx} + Bu_{yy} + Cu_{zz} + Du_{xy} + Eu_{xz} + Fu_{yz} + Gu_x + Hu_y + Ku_z + Lu = 0, \tag{2}$$

where considered. Important subclasses of families (1) and (2) are the classes of PDEs having the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = 0, \tag{3}$$

$$Au_{xx} + Bu_{yy} + Cu_{zz} + Du_{xy} + Eu_{xz} + Fu_{yz} = 0, \tag{4}$$

respectively. These include heat, Laplace's and wave equations, between others.

We also consider higher order EDPs, for example, the *bi-harmonic equation*, which is the 4^{th} order PDE

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0, \tag{5}$$

the *biwave equation*, which is the 4^{th} order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} = 0, \tag{6}$$

and the *bi-telegraph equation* (see [15]), which is the 4^{th} order PDE

$$u_{xxxx} - 2u_{xxyy} + u_{yyyy} - \lambda^4 u = 0. \tag{7}$$

For PDEs (1) and (2) we construct solutions by using the exponential function $\mathcal{E}(x) = e^{\varphi(x)}$ with respect to affine vector fields φ and defined for certain algebras. For PDEs (3) and (4) families of $\varphi\mathbb{A}$-differentiable functions have conjugate functions which are solutions. This generalizes to the well-known result that the component functions of complex analytic functions are harmonic functions.

2 Topic approach

2.1 Algebras and their fist fundamental representations

We call to a $\mathbb{R}$-linear space $\mathbb{L}$ *an algebra* if it is endowed with a bilinear product $\mathbb{L} \times \mathbb{L} \to \mathbb{L}$ denoted by $(u, v) \mapsto uv$, which is associative and commutative $u(vw) = (uv)w$ and $uv = vu$ for all $u, v, w \in \mathbb{L}$; furthermore, there exists a unit $e \in \mathbb{L}$, which satisfies $eu = u$ for all $u \in \mathbb{L}$. An element $u \in \mathbb{L}$ is called *regular* if there exists $u^{-1} \in \mathbb{L}$ called *the inverse* of u such that $u^{-1}u = e$. We also use the notation e/u for u^{-1}. If $u \in \mathbb{L}$ is not regular, then u is called *singular*. $\mathbb{L}^*$ denotes the set of all the regular elements of $\mathbb{L}$. If $u, v \in \mathbb{L}$ and v is regular, the quotient u/v means uv^{-1}.

An *algebra* $\mathbb{A}$ will be an algebra where $\mathbb{L} = \mathbb{R}^n$ and an *algebra* $\mathbb{M}$ will be an algebra where $\mathbb{L}$ is a two dimensional matrix algebra in the space of matrices $M(n, \mathbb{R})$, where the algebra product corresponds to the matrix product. We say that two matrix algebras $\mathbb{M}_1$ and $\mathbb{M}_2$ are *conjugated* if there exists an invertible matrix T such that $\mathbb{M}_1 = T\mathbb{M}_2T^{-1}$.

The $\mathbb{A}$-product between the elements of the canonical basis $\{e_1, e_2, \ldots, n\}$ of $\mathbb{R}^n$ is given by $e_ie_j = \sum_{k=1}^{n} c_{ijk}e_k$ where $c_{ijk} \in \mathbb{R}$ for $i, j, k \in \{1, 2, \ldots, n\}$ are called *structure constants* of $\mathbb{A}$. The *first fundamental representation* of $\mathbb{A}$ is the injective linear homomorphism $R : \mathbb{A} \to M(n, \mathbb{R})$ defined by $R : e_i \mapsto R_i$, where R_i is the matrix with $[R_i]_{jk} = c_{ikj}$, for $i = 1, 2, \ldots, n$.

The linear space $\mathbb{R}^2$ endowed with the product

$$\begin{array}{c|cc} \cdot & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & p_1e_1 + p_2e_2 \end{array} \tag{8}$$

is an algebra $\mathbb{A}$ which we denote by $\mathbb{A}_1(p_1, p_2)$, see [5]. These algebras are associative, commutative, and have unit $e = e_1$, see also [13]. The *first fundamental representation* of $\mathbb{A}_1(p_1, p_2)$ is defined by

$$R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(e_2) = \begin{pmatrix} 0 & p_1 \\ 1 & p_2 \end{pmatrix}. \tag{9}$$

This representation allows us to use the corresponding matrix algebra in order to get expressions of $\varphi\mathbb{A}$-functions, like the defined in the following section.

Consider three six parameters family of algebras. The linear space $\mathbb{R}^3$ endowed with the product

$$\begin{array}{c|ccc} \cdot & e_r & e_s & e_t \\ \hline e_r & e_r & e_s & e_t \\ e_s & e_s & p_7 e_r + p_1 e_s + p_2 e_t & p_8 e_r + p_3 e_s + p_4 e_t \\ e_t & e_t & p_8 e_r + p_3 e_s + p_4 e_t & p_9 e_r + p_5 e_s + p_6 e_t \end{array}, \tag{10}$$

where

$$\begin{aligned} p_7 &= p_2 p_3 + p_4^2 - p_1 p_4 - p_2 p_6, \\ p_8 &= p_2 p_5 - p_3 p_4, \\ p_9 &= p_3^2 + p_4 p_5 - p_1 p_5 - p_3 p_6, \end{aligned} \tag{11}$$

$\{e_r, e_s, e_t\} = \{e_1, e_2, e_3\}$, is an algebra, and with unit $e = e_r$, that we denote by $\mathbb{A}_r^3(p_1, \cdots, p_6)$, see [6] and [13]. The first fundamental representation of $\mathbb{A}_r^3(p_1, \cdots, p_6)$ is determined by

$$R(e_r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R(e_s) = \begin{pmatrix} 0 & p_7 & p_8 \\ 1 & p_1 & p_3 \\ 0 & p_2 & p_4 \end{pmatrix}, \quad R(e_t) = \begin{pmatrix} 0 & p_8 & p_9 \\ 0 & p_3 & p_5 \\ 1 & p_4 & p_6 \end{pmatrix}.$$

This allows us to use the corresponding matrix algebra in order to get expressions of some vector fields which are defined with this algebra product.

If $r = 1$, $s = 2$, $t = 3$, $p_1 = 0$, $p_2 = 1$, $p_3 = 0$, $p_4 = 0$, $p_5 = 1$, and $p_6 = 0$, then we call to $\mathbb{A}^1(p_1, \cdots, p_6)$ the *3D-cyclic algebra*. This appears in [16] under the name Complex numbers in three dimensions, and in [12] is used for constructing 3D harmonic functions. The matrix algebra $\mathbb{M} = R(\mathbb{A})$ is conjugated to the matrix algebra spanned by the normal form with a real simple block and a complex simple block, see [1] Section 2.2.

2.2 $\varphi\mathbb{A}$-differentiability

The pre-twisted differentiability is defined in [9], this definition is closely related with the differentiability in the sense of Lorch, see [11]. Let $\mathbb{A}$ be an algebra and φ a differentiable n-dimensional vector field in the usual sense. We say that a planar vector field $\mathcal{F}$ is $\varphi\mathbb{A}$-*differentiable (pre-twisted differentiable)* if $\mathcal{F}$ is differentiable in the usual sense and if there exists a n-dimensional vector field $\mathcal{F}'_\varphi$ which we call $\mathbb{A}$-*derivative* of $\mathcal{F}$, such that

$$d\mathcal{F}_p = \mathcal{F}'_\varphi(p)d\varphi_p, \tag{12}$$

where $\mathcal{F}'_\varphi(p)d\varphi_p(v)$ denotes the $\mathbb{A}$-product of $\mathcal{F}'_\varphi(p)$ and $\varphi_p(v)$ for every vector v in $\mathbb{R}^n$. In the same way, we say that $\mathcal{F}$ has a *second order* $\varphi\mathbb{A}$-*derivative* $\mathcal{F}''_\varphi$ if $\mathcal{F}$ is $\varphi\mathbb{A}$-differentiable, $\mathcal{F}'_\varphi$ is differentiable in the usual sense, and $\mathcal{F}''_\varphi$ is a n-dimensional vector field, such that

$$d(\mathcal{F}'_\varphi)_p = \mathcal{F}''_\varphi(p)d\varphi_p. \tag{13}$$

Therefore, in this way we define the *k-order* $\varphi\mathbb{A}$-*derivative* $\mathcal{F}^{(k)}_\varphi$ for $k \in \mathbb{N}$.

A $\varphi\mathbb{A}$-*polynomial function* $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ is defined by

$$\mathcal{P}(\xi) = c_0 + c_1\varphi(\xi) + c_2(\varphi(\xi))^2 + \cdots + c_m(\varphi(\xi))^m \tag{14}$$

where $c_0, c_1, \cdots, c_m \in \mathbb{A}$ are constants, $\xi = (x_1, \ldots, x_n)$ is $\mathbb{A}$-variable, and the products involved in $c_k(\varphi(\xi))^k$ for $k \in \{1, 2, \cdots, m\}$ are defined with respect to $\mathbb{A}$. In the same way *exponential*, *trigonometric*, and *hyperbolic* $\varphi\mathbb{A}$-*functions* are defined. If $\mathcal{P}$ and $\mathcal{Q}$ are $\varphi\mathbb{A}$-polynomial functions, the $\varphi\mathbb{A}$-*rational function* $\mathcal{P}/\mathcal{Q}$ is defined on the set $\mathcal{Q}^{-1}(\mathbb{A}^*)$. All these functions have k-order $\varphi\mathbb{A}$-derivatives for $k \in \mathbb{N}$ and the usual rules of differentiation are satisfied for this differentiability.

The $\varphi\mathbb{A}$-differentiability can be characterized by *Cauchy-Riemann equations* for $(\varphi, \mathbb{A})$ ($\varphi\mathbb{A}$-*CREs*), which is the linear system of $n(k-1)!$ PDEs obtained from

$$d\varphi(e_j)f_{u_i} = d\varphi(e_i)f_{u_j} \tag{15}$$

for $i, j \in \{1, \cdots, k\}$. For $i = 1, \cdots, k$ suppose $\varphi = (\varphi_1, \cdots, \varphi_n)$, then

$$d\varphi(e_i) = \varphi_{u_i} = \sum_{l=1}^{n} \varphi_{lu_i} e_l. \tag{16}$$

Let $f = (f_1, \cdots, f_n)$ be an $\varphi\mathbb{A}$-differentiable function. Thus, the $\varphi\mathbb{A}$-CREs are given by

$$\sum_{m=1}^{n} \sum_{l=1}^{n} (f_{mu_i}\varphi_{lu_j} - f_{mu_j}\varphi_{lu_i}) C_{lmq} = 0 \tag{17}$$

for $1 \leq i < j \leq k$ and $q = 1, \cdots, n$, which is a system of $n(k-1)!$ partial differential equations.

3 Methodology

The method proposed in this work is similar to that of Ketchum, see [7].

3.1 The Ketchum method

In [7] pp. 659, 660 P. W. Ketchum proposed functions of the form $f(w(x))$ for solving PDEs, where f is analytic in the sense of Lorch. He did not exploit this, since his conclusions were made only for the case $w(x) = x$. An algebra $\mathbb{A}$ whose analytic functions $f(w)$ satisfy Laplace's equation is called *harmonic algebra*. In the paper [14], pp. 547, it is interpreted that for an algebra to be a harmonic algebra it is required that $e_1^2 + e_2^2 + e_3^2 = 0$, but this condition is necessary for the case of $w(x, y, z) = (x, y, z)$ (in our notation $\varphi(x, y, z) = (x, y, z)$). Ketchum makes a similar claim in the introduction of [8]. This condition is used to solve PDEs, see [15].

3.2 The method for solving PDEs proposed in this project

The $\varphi\mathbb{A}$-differentiability can be used to make more explicit the method given by P. W. Ketchum in [7] for constructing solutions of PDEs of mathematical physics. Therefore, this new notion of differentiability constitutes an important key to the construction of solutions of classical mathematical physics PDEs, see [9] and [10]. In particular, families of pre-twisted differentiable functions are bi-harmonic functions.

In [10] it is shown that for the algebra $\mathbb{A}$ defined by $\mathbb{R}^3$ with the product $e_3^2 = e_2$, $e_3^2 = e_1$, where the unit e of the algebra is $e = e_1$, is a harmonic algebra by taking

$$w = \varphi(x, y, z) = \left(x + k_1, -\frac{1}{2}x + \frac{\sqrt{2}}{2}y + \frac{1}{2}z + k_2, -\frac{1}{2}x - \frac{\sqrt{2}}{2}y - \frac{1}{2}z + k_3 \right).$$

With respect to $\mathbb{A}$ one has that $e_1^2 + e_2^2 + e_3^2 = e_1 + e_3 + e_1 = (2, 0, 1)$. So, $e_1^2 + e_2^2 + e_3^2 \neq 0$. In this case $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0$.

Therefore, for solving PDE (4) the method proposed in this project requires an affine 3D vector field and an algebra $\mathbb{A}$ with respect to which

$$A\varphi_{xx} + B\varphi_{yy} + C\varphi_{zz} + D\varphi_{xy} + E\varphi_{xz} + F\varphi_{yz} = 0.$$

4 Results

4.1 Two independent variables PDEs

In the following theorem we found the algebra $\mathbb{A}$ with respect to which the components of the exponential function $\mathcal{E} = e^{\varphi}$ define solutions of the PDE (1).

Theorem 4.1 *Consider the EDP (1) and the vector field φ given by*

$$\varphi(x, y) = (ax + by + k, cx + dy + l). \tag{18}$$

Suppose that $Ac^2 + Bcd + Cd^2 \neq 0$. Thus, for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ with parameters p_1 and p_2 given by

$$p_1 = -\frac{Aa^2 + Bab + Cb^2 + Da + Eb + F}{Ac^2 + Bcd + Cd^2}, \tag{19}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd + Dc + Ed}{Ac^2 + Bcd + Cd^2}, \tag{20}$$

we have that the components f and g of the exponential function $\mathcal{E} = e^{\varphi}$ defined with respect to $\mathbb{A}$, are solutions of the PDE (1).

Example 4.1 *Consider the PDE (1) with $A = 1$, $B = 2$, $C = 3$, $D = 4$, $E = 5$, and $F = 6$. We can define φ with $c = d = 1$, and $a = b = 0$, that is, $\varphi(x, y) = (0, x + y)$. So $p_1 = -1$, $p_2 = -3/2$, and by using the first fundamental representation R we can found that f and g are given by*

$$f(x,y) = \frac{7\cos(\frac{\sqrt{7}(x+y)}{4}) + 3\sqrt{7}\sin(\frac{\sqrt{7}(x+y)}{4})}{7e^{\frac{3(x+y)}{4}}}, \tag{21}$$

and

$$g(x,y) = \frac{4\sqrt{7}\sin(\frac{\sqrt{7}(x+y)}{4})}{7e^{\frac{3(x+y)}{4}}}, \tag{22}$$

and they are solutions for (1).

If we consider the same PDE and $\varphi(x, y) = (0, x)$, so $p_1 = -6$, $p_2 = -4$,

$$f(x,y) = e^{-2x}\left(\cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)\right), \tag{23}$$

and

$$g(x,y) = \frac{\sqrt{2}}{2}e^{-2x}\sin(\sqrt{2}x). \tag{24}$$

The exponential function is given in the following propositions.

Proposition 4.1 *If ${p_2}^2 + 4p_1 < 0$, then the solutions f and g of the PDE (1) given in Theorem 4.7 for $\varphi(x, y) = (ax + by, cx + dy)$ are the following*

$$f(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\cos\left(\frac{\gamma}{2}(cx+dy)\right) - \frac{p_2}{\gamma}\sin\left(\frac{\gamma}{2}(cx+dy)\right)\right) \tag{25}$$

and

$$g(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\frac{(-{p_2}^2 - \gamma^2)\sin\left(\frac{\gamma}{2}(cx+dy)\right)}{2p_1\gamma}\right), \tag{26}$$

where $\gamma = \sqrt{-{p_2}^2 - 4p_1}$.

Proposition 4.2 *If ${p_2}^2 + 4p_1 = 0$, then the solutions f and g of the PDE (1) given in Theorem 4.7 for $\varphi(x, y) = (ax + by, cx + dy)$ are the following*

$$f(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}\left(\frac{-p_2(cx+dy)+2}{2}\right) \tag{27}$$

and

$$g(x,y) = e^{ax+by+\frac{p_2}{2}(cx+dy)}(cx+dy). \tag{28}$$

Proposition 4.3 *Let* $p_2{}^2 + 4p_1 > 0$ *and* $\gamma = \sqrt{p_2{}^2 + 4p_1}$. *Then the solutions* f *and* g *of the PDE (1) given in Theorem 4.7 for* $\varphi(x, y) = (ax + by, cx + dy)$ *are the following*

1. *If* $p_1 \neq 0$,

$$f(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma - p_2)e^{\frac{\gamma}{2}(cx+dy)} + (\gamma + p_2)e^{-\frac{\gamma}{2}(cx+dy)}}{2\gamma} \tag{29}$$

and

$$g(x, y) = e^{ax+by+\frac{p_2}{2}(cx+dy)} \frac{(\gamma^2 - p_2{}^2)e^{\frac{\gamma}{2}(cx+dy)} - (\gamma^2 - p_2{}^2)e^{-\frac{\gamma}{2}(cx+dy)}}{4p_1\gamma}. \tag{30}$$

2. *If* $p_1 = 0$,

$$f(x, y) = e^{ax+by} \tag{31}$$

and

$$g(x, y) = \frac{1}{p_2} e^{ax+by} \left(-1 + e^{p_2(cx+dy)}\right). \tag{32}$$

For PDEs of the form (3), the conjugate functions of $\varphi\mathbb{A}$-differentiable functions, define solutions.

Theorem 4.2 *Consider the PDE (3) and the affine planar vector field* φ *given in (18). Suppose that* $Ac^2 + Bcd + Cd^2 \neq 0$. *Thus, for the algebra* $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ *with parameters* p_1 *and* p_2 *given by*

$$p_1 = -\frac{Aa^2 + Bab + Cb^2}{Ac^2 + Bcd + Cd^2}, \tag{33}$$

and

$$p_2 = -\frac{2Aac + B(ad + bc) + 2Cbd}{Ac^2 + Bcd + Cd^2}, \tag{34}$$

we have that the components f *and* g *of each* $\varphi(\mathbb{A})$*-differentiable function* $\mathcal{F} = (f, g)$ *are solutions of the PDE (3).*

Theorem 4.2 is a generalization of a well known and important result, as we see in the following corollary.

Corollary 4.1 *Suppose that $A = 1$, $B = 0$, $C = 1$, and $\varphi(x, y) = (x, y)$. Then, PDE (3) is the Laplace's equation $u_{xx} + u_{yy} = 0$, and $p_1 = -1$ and $p_2 = 0$. Thus, $\mathbb{A} = \mathbb{A}_1(-1, 0)$ is the algebra of the complex numbers $\mathbb{A} = \mathbb{C}$, the $\varphi\mathbb{A}$-differentiability corresponds to the usual complex differentiability, and the components of the $\varphi\mathbb{A}$-differentiable functions are solutions of the Laplace's equation.*

Consider the 3^{th} order PDE

$$Gu_{xxx} + Hu_{xxy} + Ku_{xyy} + Lu_{yyy} = 0. \tag{35}$$

Theorem 4.3 *Consider the PDE (35), the affine planar vector field φ given in (18), and the quadratic system of equations*

$$\begin{aligned}
& (Gc^3 + Hc^2d + Kcd^2 + Ld^3)xy \\
+ \quad & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)x \\
+ \quad & Ga^3 + Ha^2b + Kab^2 + Lb^3 = 0, \\
& (Gc^3 + Hc^2d + Kcd^2 + Ld^3)y^2 + (Gc^3 + H(c^2d + bc^2) + Kcd^2 + Ld^3)x \\
+ \quad & (3Gac^2 + H(bc^2 + 2acd) + K(ad^2 + 2bcd) + 3Lbd^2)y + 3Ga^2c \\
+ \quad & 2H(abc + a^2d) + K(b^2c + 2abd) + 3Lb^2d = 0.
\end{aligned} \tag{36}$$

*If (p_1, p_2) is a solution of the quadratic system (36), for the algebra $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of all the third order $\varphi\mathbb{A}$-differentiable function, are solutions of the PDE (**??**).*

The following theorem gives solutions for the bi-harmonic PDE.

Theorem 4.4 *Consider the PDE (5), the affine planar vector field φ given in (18), and the cubic system of two equations*

$$\begin{aligned}
& (c^2 + d^2)^2(xy^2 + x^2) + 4(ac + bd)(c^2 + d^2)xy \\
& +2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)x + (a^2 + b^2)^2 = 0, \\
& (c^2 + d^2)^2(y^3 + 2xy) + 4(ac + bd)(c^2 + d^2)(y^2 + x) \\
& +2(3a^2c^2 + a^2d^2 + 4abcd + b^2c^2 + 3b^2d^2)y + 4(a^2 + b^2)(ac + bd) = 0.
\end{aligned} \tag{37}$$

If (p_1, p_2) is a solution of the cubic system (37), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of all the fourth order $\varphi\mathbb{A}$-differentiable functions are solutions of the PDE (5).

Example 4.2 *If $\varphi(x, y) = (x + y + k, x - y + l)$, then $p_1 = -1$ and $p_2 = 0$, satisfy conditions (37). Thus, for $\mathbb{A} = \mathbb{C}$ the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-harmonic equation (5). Function*

$$(x + y + k, x - y + l)^{-1} = \left(\frac{x + y + k}{(x + y + k)^2 + (x - y + l)^2}, \frac{x - y + l}{(x + y + k)^2 + (x - y + l)^2}\right),$$

has components

$$f(x, y) = \frac{x + y + k}{(x + y + k)^2 + (x - y + l)^2}, \qquad g(x, y) = \frac{x - y + l}{(x + y + k)^2 + (x - y + l)^2},$$

which are bi-harmonic functions. Also function $(x + y + k, x - y + l)^4$, has components

$$\begin{aligned}
f(x, y) &= k^4 + l^4 - 6k^2l^2 - 4x^4 - 8kx^3 - 8lx^3 - 24klx^2 + 4k^3x + 4l^3x - 12kl^2x - 12k^2lx \\
&\quad -4y^4 - 8ky^3 + 8ly^3 + 24kly^2 + 24x^2y^2 + 24kxy^2 + 24lxy^2 + 4k^3y - 4l^3y \\
&\quad -12kl^2y + 12k^2ly + 24kx^2y - 24lx^2y + 24k^2xy - 24l^2xy, \\
g(x, y) &= -4kl^3 + 4k^3l + 8kx^3 - 8lx^3 + 12k^2x^2 - 12l^2x^2 + 4k^3x - 4l^3x - 12kl^2x + 12k^2lx \\
&\quad -8ky^3 - 8ly^3 - 16xy^3 - 12k^2y^2 + 12l^2y^2 - 24kxy^2 + 24lxy^2 - 4k^3y - 4l^3y \\
&\quad +12kl^2y + 12k^2ly + 16x^3y + 24kx^2y + 24lx^2y + 48klxy,
\end{aligned}$$

which are bi-harmonic functions.

The following theorem gives solutions for the bi-wave PDE.

Theorem 4.5 *Consider the PDE (6), the affine planar vector field φ given in (18), and the cubic system of two equations*

$$\begin{gathered}
(c^2 - d^2)^2(xy^2 + x^2) + 4(ac - bd)(c^2 - d^2)xy \\
+2(3a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + 3b^2d^2)x + (a^2 - b^2)^2 = 0, \\
(c^2 - d^2)^2(y^3 + 2xy) + 4(ac - bd)(c^2 - d^2)(y^2 + x) \\
+2(3a^2c^2 - a^2d^2 - 4abcd - b^2c^2 + 3b^2d^2)y + 4(a^2 - b^2)(ac - bd) = 0.
\end{gathered} \tag{38}$$

If (p_1, p_2) is a solution of the cubic system (38), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of all the fourth order $\varphi\mathbb{A}$-differentiable functions are solutions of the bi-wave equation (6).

The following theorem gives solutions for the bi-telegraphic PDE.

Theorem 4.6 *Consider the PDE (7), the affine planar vector field φ given in (18), and the cubic system of two equations*

$$\begin{gathered}(c^2-d^2)^2(xy^2+x^2)+4(ac-bd)(c^2-d^2)xy\\ +2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)x+(a^2-b^2)^2-\lambda^4=0,\\ (c^2-d^2)^2(y^3+2xy)+4(ac-bd)(c^2-d^2)(y^2+x)\\ +2(3a^2c^2-a^2d^2-4abcd-b^2c^2+3b^2d^2)y+4(a^2-b^2)(ac-bd)=0.\end{gathered} \tag{39}$$

If (p_1, p_2) is a solution of the cubic system (39), then for $\mathbb{A} = \mathbb{A}_1(p_1, p_2)$ the components of the functions

$$e^{\varphi(x,y)}, \quad \sin(\varphi(x,y)), \quad \cos(\varphi(x,y)), \quad \sinh(\varphi(x,y)), \quad \cosh(\varphi(x,y)),$$

are solutions of the bi-telegraphic equation (7).

4.2 Three independent variables PDEs

Given a PDE of the form (2) we look for a 3D affine vector field

$$\varphi(x,y,z) = (a_1x+b_1y+c_1z+k_1, a_2x+b_2y+c_2z+k_2, a_3x+b_3y+c_3z+k_3), \tag{40}$$

and an algebra $\mathbb{A} = \mathbb{A}_1^3(p_1, \ldots, p_6)$ such that the components of the $\varphi\mathbb{A}$-exponential function

$$\mathcal{E}(x,y,z) = e^{\varphi(x,y,z)} \tag{41}$$

are solutions of the given PDE. We give a system of three quartic equations of fifteen variables for which every solution determines a 3D affine vector field and an algebra $\mathbb{A}$ such that columns of function (41) with respect to $\mathbb{A}$, define solutions of the given PDE. For some PDEs we can use sin, cos, sinh, and cosh functions instead of the exponential function.

Consider equalities

$$\begin{aligned} A\varphi_x^2 + B\varphi_y^2 + C\varphi_z^2 + D\varphi_x\varphi_y + E\varphi_x\varphi_z + F\varphi_y\varphi_z \\ +G\varphi_x + H\varphi_y + K\varphi_z + L(1,0,0) = 0, \end{aligned} \tag{42}$$

$$A\varphi_x^2 + B\varphi_y^2 + C\varphi_z^2 + D\varphi_x\varphi_y + E\varphi_x\varphi_z + F\varphi_y\varphi_z + L(1,0,0) = 0, \tag{43}$$

$$A\varphi_x^2 + B\varphi_y^2 + C\varphi_z^2 + D\varphi_x\varphi_y + E\varphi_x\varphi_z + F\varphi_y\varphi_z - L(1,0,0) = 0, \tag{44}$$

$$A\varphi_x^2 + B\varphi_y^2 + C\varphi_z^2 + D\varphi_x\varphi_y + E\varphi_x\varphi_z + F\varphi_y\varphi_z = 0. \tag{45}$$

The equalities (42), (43), (44), and (45) give rise to solutions of a PDEs, as we see in the following theorem.

Theorem 4.7 *Let $\mathbb{A}$ be a 3D algebra and the affine 3D vector field (40).*

1) If equation (42) is satisfied, then the components of $\mathcal{E} = e^{\varphi}$ are solutions for PDE (2).

2) If equation (43) is satisfied, then the components of the functions $\mathcal{S} = sin \circ \varphi$ and $\mathcal{C} = cos \circ \varphi$ are solutions for PDE (2) with $G = 0$, $H = 0$, and $K = 0$.

3) If equation (44) is satisfied, then the components of $\mathcal{SH} = sinh \circ \varphi$ and $\mathcal{CH} = cosh \circ \varphi$ are solutions for PDE (2) with $G = 0$, $H = 0$, and $K = 0$.

4) If equation (45) is satisfied, then the components of all the $\varphi\mathbb{A}$-differentiable functions are solutions for (4).

In the following corollary we give a 3D linear vector field φ such that components of all the $\varphi\mathbb{A}$-differentiable functions with respect to the cyclic algebra, are harmonic functions.

Corollary 4.2 *For the 3D cyclic algebra $\mathbb{A} = \mathbb{A}^1(0,1,0,0,1,0)$ and the 3D vector field*

$$\varphi(x,y,z) = \left(x + k_1, -\frac{1}{2}x + \frac{\sqrt{2}}{2}y + \frac{1}{2}z + k_2, -\frac{1}{2}x - \frac{\sqrt{2}}{2}y - \frac{1}{2}z + k_3 \right), \tag{46}$$

the components of the $\varphi\mathbb{A}$-differentiable functions are solutions of the 3D Laplace's equation.

Complex differentiability of a planar vector field V is equivalent to the differentiability of V in the usual sense and the component functions of V satisfy the Cauchy-Riemann equations. Thus, complex differentiability is characterized by the Cauchy-Riemann equations. In the same way, the pre-twisted Cauchy-Riemann equations characterize the $\varphi\mathbb{A}$-differentiability.

Corollary 4.3 *Let $\mathcal{F} = (u, v, w)$ be a differentiable vector field with component functions satisfying the pre-twisted Cauchy-Riemann equations*

$$\begin{aligned} -\sqrt{2}v_x + \sqrt{2}w_x &= 2u_y - v_y - w_y & \quad -v_y + w_y &= 2u_z - v_z - w_y z \\ -\sqrt{2}u_x + \sqrt{2}v_x &= -u_y - v_y + 2w_y\ , & \quad -u_y + v_y &= -u_z - v_z + 2 - w_z\ , \\ \sqrt{2}u_x - \sqrt{2}w_x &= -u_y + 2v_y - w_y & \quad -u_y - w_y &= -u_z + 2v_z - w_z \end{aligned} \tag{47}$$

then u, v, and w satisfy the 3D Laplace's equation.

Example 4.3 *Consider the cyclic algebra $\mathbb{A}^1(0,1,0,0,1,0)$ and the 3D vector field φ defined in (46) with $k_i = 0$ for $i = 1,2,3$. Since*

$$\begin{aligned} \varphi(x,y,z)^2 &= \frac{3x^2 - 2y^2 - z^2 - 2\sqrt{2}yz}{2}e_1 \\ &+ \frac{-3x^2 + 2y^2 + z^2 + 6\sqrt{2}xy + 6xz + 2\sqrt{2}yz}{4}e_2 \\ &+ \frac{-3x^2 + 2y^2 + z^2 - 6\sqrt{2}xy - 6xz + 2\sqrt{2}yz}{4}e_3, \end{aligned}$$

by Corollary 4.2, functions

$$u(x,y,z) = \frac{3x^2 - 2y^2 - z^2 - 2\sqrt{2}yz}{2}, \tag{48}$$

$$v(x,y,z) = \frac{-3x^2 + 2y^2 + z^2 + 6\sqrt{2}xy + 6xz + 2\sqrt{2}yz}{4}, \tag{49}$$

$$w(x,y,z) = \frac{-3x^2 + 2y^2 + z^2 - 6\sqrt{2}xy - 6xz + 2\sqrt{2}yz}{4}, \tag{50}$$

are solutions for the 3D Laplace's equation.

Example 4.4 *Consider the cyclic algebra $\mathbb{A}^1(0,1,0,0,1,0)$ and the 3D vector field φ defined in (46) with $k_i = 0$ for $i = 1,2,3$. Since*

$$\frac{\varphi(x,y,z)}{(1,2,3) + \varphi(x,y,z)} = \frac{p(x,y,z)}{s(x,y,z)}e_1 + \frac{q(x,y,z)}{s(x,y,z)}e_2 + \frac{r(x,y,z)}{s(x,y,z)}e_3,$$

where

$$
\begin{aligned}
p(x,y,z) &= 9x^4+36x^3-156x^2+192x+4y^4+12x^2y^2+24xy^2-40y^2+48\sqrt{2}xy \\
&\quad -48\sqrt{2}y+z^4-4z^3+6x^2z^2+12xz^2-4y^2z^2-12z^2-12x^2z-24xz \\
&\quad +8y^2z-48\sqrt{2}xyz+48\sqrt{2}yz+32z-80, \\
q(x,y,z) &= 9x^4-72x^3+204x^2-240x+4y^4-24\sqrt{2}y^3+12x^2y^2-48xy^2+8y^2 \\
&\quad -36\sqrt{2}x^2y+48\sqrt{2}xy+z^4-16z^3+6x^2z^2-24xz^2-4y^2z^2+12\sqrt{2}yz^2 \\
&\quad +60z^2-48x^2z+120xz+32y^2z+24\sqrt{2}xyz-48\sqrt{2}yz-112z+112, \\
r(x,y,z) &= 9x^4-72x^3+132x^2-96x+4y^4-24\sqrt{2}y^3+12x^2y^2-48xy^2-56y^2 \\
&\quad +36\sqrt{2}x^2y-96\sqrt{2}xy+48\sqrt{2}y+z^4+8z^3+6x^2z^2-24xz^2-4y^2z^2 \\
&\quad -12\sqrt{2}yz^2-12z^2+24x^2z-24xz-16y^2z+24\sqrt{2}xyz+32z+16, \\
s(x,y,z) &= 162x^4-648x^3+1080x^2-864x-72y^4+216x^2y^2-432xy^2+144y^2 \\
&\quad +18z^4-72z^3+108x^2z^2-216xz^2-72y^2z^2+216z^2-216x^2z \\
&\quad +432xz+144y^2z-288z+280,
\end{aligned} \tag{51}
$$

by Corollary 4.2, functions

$$
u(x,y,x)=\frac{p(x,y,z)}{s(x,y,z)}, \qquad v(x,y,z)=\frac{q(x,y,z)}{s(x,y,z)}, \qquad w(x,y,z)=\frac{r(x,y,z)}{s(x,y,z)}, \tag{52}
$$

are solutions for the 3D Laplace's equation, where p, q, r, and s are the functions given in (51).

5 Conclusiones

The proposed method is very fruitful, can be solved familiar from partial differential equations of mathematical physics, and can be generalized:

1. The order of the PDEs considered can be increased.

2. The number of dependent variables of the PDEs under consideration can be increased.

3. The order and the number of variables of the PDEs considered can be increased.

The bi-harmonic equation arises from the continuum mechanics, including linear elasticity theory and the solution of Stokes flows. The method given in this project allows the construction of families of solutions of the bi-harmonic equation. Therefore, our method has an important relevance in physics-mathematics PDEs.

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