New Astronomy 82 (2021) 101458

Contents lists available at ScienceDirect

New Astronomy

journal homepage: www.elsevier.com/locate/newast

Exact analytic solution of an unsolvable class of first Lane–Emden equation for polytropic gas sphere

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ARTICLE INFO

Keywords:
Analytical solution
Boundary conditions
Polytropic gas
Ultra-relativistic gas
Adiabatic star
Exoplanets

ABSTRACT

This article provides for the first time a general analytical solution to the Lane-Emden equation of the first kind. So far only three known analytical solutions are found in the literature, for the following values of **n**: 0, 1 and 5. A common feature these three solutions share is their boundary conditions: $\theta(\xi)|_{\xi=0} = 1$ and $\frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0$. If a third boundary condition $\frac{d^2\theta(\xi)}{d\xi^2}|_{\xi=0}, = -1$ is used, only the solution for $n = 1$ is able to meet all three. In order to address this difference, our solution aims to be more inclusive and takes into account $\theta(\xi) = \frac{1}{\xi}$ and the constant solution. By keeping τ in parametric form, we found out that $\theta(\xi(\tau)) = \frac{1}{\xi(\tau)} \to 1$ when $\xi \to 0$. Thus proving that $\frac{1}{\xi} \to 1$ in the origin. It is worth noting that upon integrating the Lane-Emden equation, we came across five parameters. Three of them depend on the three boundary conditions used and two can be adjusted numerically.

In order to demonstrate the validity of our solution, we tested it on six cases of interest to the scientific community related to studies on real stars and exoplanets. The adiabatic exponents are $n = 1.5$, $n = 2$, $n = 2.592$, $n = 3$, $n = 3.23$ and $n \simeq 5$ contained in the intervals $1 < n < 5$ and $5 \lesssim n < 9$. It is worth noting that four of these cases are of particular importance; $n = 1.5$, which corresponds to an adiabatic star supported by the pressure of non-relativistic gas; $n = 3$, which corresponds to an adiabatic star supported by the pressure of an ultra-relativistic gas. Finally, $n = 2.592$ and $n = 3.23$, which correspond to exoplanets. The obtained solution of the Lane–Emden equation of the first kind proves valid for any value of **n**.

1. Introduction

As far as the mathematical aspect behind this work goes, the applicability of the Lane–Emden equation of the first kind simply cannot be overstated. Problems that cannot be solved exactly, such as those defined on infinite or semi-infinite intervals, are traditionally addressed using semi-analytical or numerical approximation methods (e.g. the Adomian decomposition method or the Homotopy perturbation method, see (Parand et al., 2017). Every field, ranging from astrophysics and fluid dynamics to quantum mechanics, has problems like these.

The physical aspect of the first-kind Lane–Emden equation involves the analysis of different phenomena of equilibrium for non rotating fluids of self-gravitating stars. Although limited in precision, existing models are accurate enough to model stellar structures. A minor influence over stars is required to reduce the effects that break up the rotational symmetry, such as convections, magnetic fields B, and other existing physical phenomena. It is implied non stationary states occur in gasses (for example star's pulsation) when there is a great influence from rotational effect and / or a strong magnetic field B.

The very same effects that influence the system's spherical symmetry are the very same conditions used to model the non-linear and differential Lane-Emden equation of the first kind of index n. Gasses become stable in the $\frac{6}{5} < \gamma < \frac{4}{3}$ interval, which is exactly where the hydrostatic model is applied.

Concrete examples involve, for example, exoplanet systems, or the individual exoplanets among the thousands discovered in the last twenty years or so. For the former, see $HD10180$, $Kepler-32$, $Kepler-33$, $Kepler102$, and $Kepler-186$, see Geroyannis (2015). For the latter, those seven Earth-sized planets transiting the ultracool dwarf star TRAPPIST-1, see (Lingam and Loeb, 2018) and Geroyannis (2017). Other scenarios apply for stellar structures and polytropic gasses as well. For instance, studies involving the evolution of compact binary stellar systems with mass-losing secondary masses that range between $\sim 0.01M_\odot$ and $\sim 1M_\odot$ ($M_\odot$ representing solar mass), see Rappaport et al. (1983). Another example includes (Jang, 2013), who studied the nonlinear instability modeled by the Euler-Poisson system, which applies to polytropic gasses. The solution herein described

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https://doi.org/10.1016/j.newast.2020.101458
Received 18 March 2020; Received in revised form 25 June 2020; Accepted 23 July 2020
Available online 24 July 2020
1384-1076/

complements the findings that have made use of numerical aproximation methods.

The equation itself is usually expressed as $\frac{1}{\xi^2}\frac{d(\xi^2\frac{d\theta(\xi)}{d\xi})}{d\xi} = -\theta^n(\xi)$, where $\theta(\xi)$ represents the polytropic star's temperature and ξ its dimensionless radius. Previous solutions for this equation exist. They are limited to the boundary conditions $\theta(\xi = 0) = 1$, representing the system's central density, and $\frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0$, which proves there is no mass inside a radius that equals zero. We can go even further and define a third boundary condition given by $\frac{d^2\theta(\xi)}{d\xi^2}|_{\xi=0} = -1$. The rational behind it is that since $\theta(\xi)$ monotonically decreases as ξ increases –and eventually vanishes at point ξ_0 (i.e. $\theta(\xi_0) = 0$) – ξ_0 is the first root of θ, see Geroyannis (1993). What this means is that the second derivative represents the system's mass change rate. When positive, the system's mass increases or decreases quicker; when negative, it increases or decreases slower.

By adjusting two parameters, *c* and *k*, numerically, and taking into account all three boundary conditions, we obtain an analytical solution that applies to any value of *n*; not to mention that the parameters themselves depend on *n* as well. Our solution is validated by testing it on four cases of particular importance: $n = 1.5$ and 3 (see Rappaport et al. (1983)), for stars, and $n = 2.592$ and 3.23, for exoplanets. $n = 1.5$ corresponds to an adiabatic star supported by the pressure of a non-relativistic gas (fully convective star, as a *M, L*, or *T* dwarf for $n = 0$ to 1.5). $n = 3$ corresponds to an adiabatic star supported by the pressure of an ultra-relativistic gas (fully radiative star). $n = 2.592$ corresponds to an exoplanet, see Geroyannis (2015), detected via a variety of techniques, see Kane and Gelino (2014).

This paper is organized as follows. In section 2, we present the exact solutions for the Lane-Emden equation of the first kind. In section 3, the solutions validation of the Lane-Emden equation. Finally, in section 4 we present the conclusions.

2. Theoretical analysis; first-kind analytical solution

As stated by Böhmer and Harko (2009), the Lane-Emden equation of the first kind is as follows:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0 \tag{1}$$

where $\theta(\xi)$ is related to the density by means of the definition $\rho = \rho_c\theta^n(\xi)$, and n is called the polytropic index, subject to the boundary conditions $\theta(\xi)|_{\xi=0} = 1$, $\frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0$ and $\frac{d^2\theta(\xi)}{d\xi^2}|_{\xi=0}, = -1$. In order to further prove the validity of our analytical solution, the following boundary conditions were also used: $\theta(\xi)|_{\xi=0} = 1$, $\frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0$ and $\theta(\xi)|_{\xi=\xi_0} = 0$ where ξ_0 is the first root of $\theta(\xi)$.

Currently, three exact solutions to Eq. (1) exist, for $n = 0, 1$ and 5, as shown next (respectively): $\theta(\xi) = 1 - \frac{\xi^2}{6}$, $\theta(\xi) = \frac{\sin(\xi)}{\xi}$ and $\theta(\xi) = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$ (see Bender et al. (1989)). Only $\theta(\xi) = \frac{\sin(\xi)}{\xi}$ satisfies all three initial conditions. Our solution applies for any value of *n*.

In order to extend our solution from the initial values of *n* to just about any value, we proceed as follows. Substituing $\xi = \exp(t)$ as well as $\theta(t) = w(t)\exp(\alpha t)$ in Eq. (1), with $\alpha = \frac{2}{1-n}$ (which is only valid for $n = 3$, as shown below, see Eqs. (8) and (9)) leads to Eq. (1) becoming:

$$\frac{d^2w(t)}{dt^2} + \frac{n-5}{n-1}\frac{dw(t)}{dt} + \frac{2(3-n)}{(n-1)^2}w(t) + w^n(t) = 0 \tag{2}$$

for $n \neq 1$, see Böhmer and Harko (2009).

2.1. Proof of equation (2)

Substituting $\xi = \exp(t)$ into Eq. (1), we obtain:

$$\frac{d^2\theta(t)}{dt^2} + \frac{d\theta(t)}{dt} + \theta^n(t) = 0 \tag{3}$$

by stating that $\theta(t) = w(t)\exp(\alpha t)$, Eq. (3) is transformed to:

$$\frac{d^2w(t)}{dt^2} + (2\alpha+1)\frac{dw(t)}{dt} + (\alpha^2+\alpha)w(t) + w^n(t)\exp((n\alpha - \alpha + 2)) = 0 \tag{4}$$

From Eq. (3), one can obtain $n\alpha - \alpha + 2 = 0$; therefore, $\alpha = \frac{2}{1-n}$. Eq. (4) is then transformed to:

$$\frac{d^2w(t)}{dt^2} + \frac{n-5}{n-1}\frac{dw(t)}{dt} + \frac{2(3-n)}{(n-1)^2}w(t) + w^n(t) = 0 \tag{5}$$

by considering $\theta(t) = w(t)\exp(\mp\alpha t)$ and $\xi = \exp(t)$, the solution of Eq. (3) for the interval $0 < n < 1$ so long as $\theta(t) = w(t)\exp(-\alpha t)$ is:

$$\theta(\xi(\tau)) = \theta_H(\xi(\tau)) + \theta_{IN}(\xi(\tau)) = \theta(\tau) = a + \frac{b}{\xi(\tau)} + d\tau\xi^{\frac{2}{n-1}}(\tau) \tag{6}$$

The following cases must meet $\theta(t) = w(t)\exp(\alpha t)$, then is obtained

$$\theta(\xi(\tau)) = \theta_H(\xi(\tau)) + \theta_{IN}(\xi(\tau)) = \theta(\tau) = a + \frac{b}{\xi(\tau)} + d\tau\xi^{\frac{2}{1-n}}(\tau) \tag{7}$$

and are valid for interval $1 < n < 5$:

$$\theta(\xi(\tau)) = \theta_H(\xi(\tau)) + \theta_{IN}(\xi(\tau)) = \theta(\tau) = a + \frac{b}{\xi(\tau)} + d\tau\xi^{\frac{2}{n-5}}(\tau) \tag{8}$$

and for interval $5 \lesssim n < 9$:

$$\theta(\xi(\tau)) = \theta_H(\xi(\tau)) + \theta_{IN}(\xi(\tau)) = \theta(\tau) = a + \frac{b}{\xi(\tau)} + d\tau\xi^{\frac{4}{n-9}}(\tau) \tag{9}$$

2.2. Proof of the transformation

Using the chain rule in order to obtain the second derivative of $\theta(\xi(\tau))$, the following is stated:

$$\frac{d^2\theta(\xi(\tau))}{d\xi^2(\tau)} = -\frac{\tau_\xi\xi_{\tau\tau}\theta_\tau}{\xi_\tau^2} + \frac{\tau_\xi\theta_{\tau\tau}}{\xi_\tau} \text{ and } \frac{d\theta(\xi(\tau))}{d\xi(\tau)} = \tau_\xi\theta_\tau \tag{10}$$

and using the results of Eq. (32), given by $\xi_\tau(\tau) = \frac{d\xi(\tau)}{d\tau} = \frac{2\xi(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})} \equiv y(\xi(\tau))$, then $\xi_{\tau\tau}(\tau) = y_\tau(\xi(\tau))$. A specific case, when $\alpha = \frac{2}{1-n}$, should satisfy Eq. (1)and it is obtained from Eq. (7), more especifically by $\theta(\xi(\tau)) = d\tau\xi^\alpha$ (*d* is an arbitrary constant); then, by deriving with respect to τ, the following expression is obtained:

$$\theta_\tau(\xi(\tau)) = \xi^\alpha + \alpha\xi^{\alpha-1}\tau\xi_\tau \text{ and } \theta_{\tau\tau}(\xi(\tau)) = 2\alpha\xi^{\alpha-1}\xi_\tau + \alpha(\alpha-1)\xi^{\alpha-2}\tau\xi_\tau^2 + \alpha\xi^{\alpha-1}\tau\xi_{\tau\tau} \tag{11}$$

by substituing Eq. (11) into Eq. (10), the following expression is obtained:

$$\theta_{\xi\xi}(\xi(\tau)) = \left[-\frac{\xi_{\tau\tau}}{\xi_\tau^3} + 2\alpha\frac{\xi^{-1}}{\xi_\tau} + \alpha(\alpha-1)\xi^{-2}\tau\right]\xi^\alpha \text{ and } \frac{2}{\xi}\theta_\xi(\xi(\tau)) = \left[\frac{2}{\xi\xi_\tau} + \frac{2\alpha\tau}{\xi^2}\right]\xi^\alpha \tag{12}$$

and by substituing in $\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0$, and now by using $\xi_\tau(\tau) = y(\xi(\tau))$ and $\xi_{\tau\tau}(\tau) = y_\tau(\xi(\tau))$, and by substituing them into of Eq. (12), the following is obtained:

$$-\left[\frac{y_\tau}{y^3} + 2\alpha\frac{\xi^{-1}}{y} + \alpha(\alpha-1)\xi^{-2}\tau + 2\frac{\xi^{-1}}{y} + 2\alpha\xi^{-2}\tau\right]\xi^\alpha = -\theta^n(\xi) \tag{13}$$

let $y = \frac{1}{w}$, and by substituting it into Eq. (13), the next expression is obtained:

$$[ww_\tau + 2(\alpha+1)\xi^{-1}w]\xi^\alpha = -\alpha(\alpha+1)\xi^{-2+\alpha}\tau - \theta^n(\xi) \tag{14}$$

for $\theta^n(\xi) = (\tau\xi^\alpha)^n$, and by equating to expression (14), the following is obtained:

$$[ww_\tau + 2(\alpha+1)\xi^{-1}w]\xi^\alpha = -\alpha(\alpha+1)\xi^{-2+\alpha}\tau - \tau^n\xi^{\alpha n} \tag{15}$$

and the next expression is obtained:

$$ww_\tau + 2(\alpha+1)\xi^{-1}w = -\alpha(\alpha+1)\xi^{-2}\tau - (n)\tau^n\xi^{-2} \tag{16}$$

by applying the transformation $\xi w = v$ to Eq. (16), the following is obtained:

$$vv_\tau + (2\alpha+1)v = -\alpha(\alpha+1)\tau - \tau^n \tag{17}$$

where Eq. (17) is an Abel-family differential equation, see Panayotounkos (2005) and Polyanin and Zaitsev (1999).

By substituting $v = \frac{d\tau}{dt}$ in Eq. (17):

$$\frac{d\tau}{dt}\frac{d\frac{d\tau}{dt}}{d\tau} + (2\alpha+1)\frac{d\tau}{dt} = -\alpha(\alpha+1)\tau - \tau^n \tag{18}$$

then Eq. (18) is changed to:

$$\frac{d^2\tau}{dt^2} + (2\alpha+1)\frac{d\tau}{dt} + \alpha(\alpha+1)\tau = -\tau^n \tag{19}$$

and $t = \log(\xi)$, applied to Eq. (19), produces:

$$\xi^2\frac{d^2\tau}{d\xi^2} + (2\alpha+2)\xi\frac{d\tau}{d\xi} + \alpha(\alpha+1)\tau = -\tau^n \tag{20}$$

now, let the transformation be $\tau = \theta(\xi)\xi^{-\alpha}$ and substituting into Eq. (20):

$$\text{it is reduced to } \frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta^n(\xi) = 0 \tag{21}$$

which is the spherical Lane–Emden equation of the first kind.

3. Solutions validation

The analytical solution of the Lane-Emden equation of the first kind is found in the parametric form. That is, the solution of Eq. (1) is provided by $\theta(\xi(\tau)) = \theta(\tau)$, where τ is a dummy variable. Thus the analytical solution, Eqs. (8) and (9), are determined by the parameters *a, b* and *d*, which are arbitrary constants (see Tables 1, 2). We turn our attention to determining the influence of the three model parameters on the solution profiles, which depends of the value of *n* and the boundary conditions given by Eqs. (22) and (23):

$$\theta(\xi)|_{\xi=0} = 1,\ \frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0 \text{ and } \frac{d^2\theta(\xi)}{d\xi^2}|_{\xi=0} = -1 \tag{22}$$

and/or

$$\theta(\xi)|_{\xi=0} = 1,\ \frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0 \text{ and } \theta(\xi)|_{\xi=\xi_0} = 0, \tag{23}$$

where ξ_0 is the first root of $\theta(\xi)$.

3.1. Case i for $1 < n < 5$

Let $w(t) = \tau_1$ in Eq. (2), and $\theta(t_1) = w(t_1)\exp(\alpha t_1) = \tau_1\exp(\frac{2t_1}{1-n})$ and considering $\frac{d\tau_1(t_1)}{dt} = v_1$, then Eq. (2) changes to:

$$\frac{d^2\tau_1}{dt^2} - \frac{5-n}{n-1}\frac{d\tau_1}{dt} = \frac{dv_1}{dt} - \frac{5-n}{n-1}v_1 = 2\frac{n-3}{(n-1)^2}\tau_1 - \tau_1^n \tag{24}$$

now, using the chain rule in $\frac{d\tau_1}{dt_1} = \frac{dw}{dw}\frac{d\tau_1}{dt_1} = \tau_1\frac{d\tau_1}{dw}$, and substiting it into Eq. (24) (see Eq. (17)):

$$\begin{aligned} v_1\frac{dv_1}{d(\frac{n-1}{5-n})\tau_1} - v_1 &= \frac{n-1}{5-n}\left[2\frac{n-3}{(n-1)^2}\tau_1 - \tau_1^n\right] = v_1\frac{dv_1}{d\tau_2} - v_1 \\ &= 2\frac{n-3}{(5-n)^2}\tau_1 - \left[\frac{n-1}{5-n}\right]^{n+1}\tau_1^n \end{aligned} \tag{25}$$

where $(\frac{n-1}{n-5})\tau_1 = \tau_2$, which happens to be incomplete. It ought to be:

$$v_1\frac{dv_1}{d\tau_2} - v_1 = 2\frac{n-3}{(5-n)^2}\tau_2 - \left[\frac{n-1}{5-n}\right]^{n+1}\frac{\tau_2^n}{n} \tag{26}$$

This happens to be an Abel equation. $\frac{1}{n}$ was obtained by adjusting the analytical solution $\Theta(\xi) = \frac{1}{\frac{1}{nn-1}}\tau\xi^\alpha$, Eq. (1), with numerical solutions reported in the literature. The solution of Eq. (26) is:

$$v_1 = \frac{d\tau_1}{dt} = \frac{1}{2}(\tau_2 + c)(R(\tau_2) + \frac{1}{3}) \tag{27}$$

(Panayotounkos, 2005).

Now, considering $t_1 = \ln(\xi_1)$, Eq. (27) changes to:

$$(\frac{n-1}{5-n})\frac{d\tau_1}{d\ln(\xi_1)} = \frac{d\tau_2}{d\ln(\xi_1)} = \frac{1}{2}(\frac{n-1}{5-n})(\tau_2 + c)(R(\tau_2) + \frac{1}{3}) \tag{28}$$

then, by solving Eq. (28), it is obtained:

$$\xi_1 = \left[\exp\int(\frac{2d\tau_2}{(\tau_2+c)(R(\tau_2)+\frac{1}{3})})\right]^{\frac{n-1}{5-n}} = \xi^{\frac{n-1}{5-n}} \tag{29}$$

where:

$$\xi = \exp\int(\frac{2d\tau_2}{(\tau_2+c)(R(\tau_2)+\frac{1}{3})}) \tag{30}$$

Eq. (30) represents a straight line, as plotted below in Fig. 1.

3.1.1. Proof that equation (30) is a straight line

The equality

$$\theta(\xi(\tau)) = \xi(\tau) \tag{31}$$

represents a straight line, where Eq. (31) must satisfy $\frac{d^2\xi(\tau)}{d\xi^2} = 0$ (assuming $\tau = \tau_2$). By substituing Eq. (30) into Equation $\frac{d^2\xi(\tau)}{d\xi^2}$, we get:

$$\begin{aligned} &\xi_\tau(\tau) = \frac{d\xi(\tau)}{d\tau} = \frac{2\xi(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})} \text{ or } \tau_\xi = \frac{d\tau}{d\xi} = \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi(\tau)} \text{ then} \\ &\xi_{\tau\tau}(\tau) = \frac{d^2\xi(\tau)}{d\tau^2} = \frac{2\xi_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)}{(\tau+c)^2(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)R_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})^2} \\ &\qquad \tau_{\xi\xi} = \frac{\tau_\xi(R(\tau)+\frac{1}{3}) + (\tau+c)R_\xi(\tau)}{2\xi(\tau)} - \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi^2(\tau)} \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\theta_\xi(\xi(\tau)) = \frac{d\theta(\xi(\tau))}{d\xi} = \frac{d\theta(\xi(\tau))}{d\tau}\frac{d\tau}{d\xi} = \theta_\tau(\xi(\tau))\tau_\xi \text{ then} \\ &\frac{d^2\theta(\xi(\tau))}{d\xi^2} = \frac{d^2\tau}{d\xi^2}\frac{d\theta(\xi(\tau))}{d\tau} + (\frac{d\tau}{d\xi})^2\frac{d^2\theta(\xi(\tau))}{d\tau^2} = \tau_{\xi\xi}\theta_\tau(\xi(\tau)) + \tau_\xi^2\varphi_{\tau\tau}(\xi(\tau)) \end{aligned} \tag{33}$$

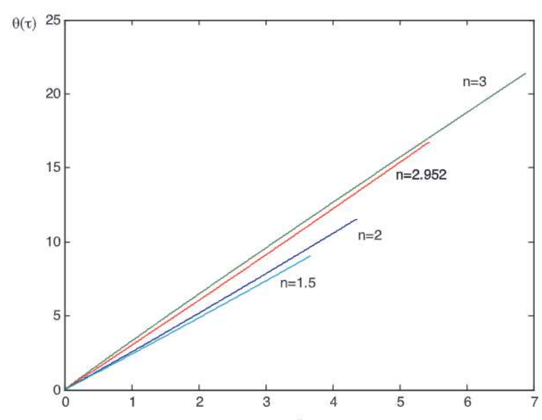

Fig. 1. Asymptotic Linear behavior of Lane–Emden equation of the first kind. For $n = 1.5$, 2, 2.592 and 3. in Eq. (30), for $\xi(\tau)$, which must represent a straight line.

By substituing in $\frac{d^2\xi(\tau)}{d\xi^2}$ the results of Eqs. (32) and (33) and $\theta(\xi(\tau)) = \xi(\tau)$, we obtain:

$$\frac{d^2\theta(\xi(\tau))}{d\xi^2} = \left[\frac{\tau_\xi(R(\tau)+\frac{1}{3})+(\tau+c)R_\xi(\tau)}{2\xi(\tau)} - \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi^2(\tau)}\right]\xi_\tau(\tau)+$$
$$+\tau_\xi^2\left[\frac{2\xi_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)}{(\tau+c)^2(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)R_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})^2}\right]=$$
$$=\left[\frac{\tau_\xi(R(\tau)+\frac{1}{3})}{2\xi(\tau)} + \frac{(\tau+c)R_\xi(\tau)}{2\xi(\tau)} - \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi^2(\tau)}\right]\frac{1}{\tau_\xi}+$$
$$+\tau_r^2\left[\frac{2\xi_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)}{(\tau+c)^2(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)R_\tau(\tau)}{(\tau+c)(R(\tau)+\frac{1}{3})^2}\right]=$$
$$=\frac{(R(\tau)+\frac{1}{3})}{2\xi(\tau)} + \frac{(\tau+c)R_\tau(\tau)}{2\xi(\tau)\tau_\xi} - \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi^2(\tau)\tau_\xi}+$$
$$+\frac{2\xi_\tau(\tau)\tau_\xi^2}{(\tau+c)(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)\tau_\xi^2}{(\tau+c)^2(R(\tau)+\frac{1}{3})} - \frac{2\xi(\tau)R_\tau(\tau)\tau_\xi^2}{(\tau+c)(R(\tau)+\frac{1}{3})^2}=$$
$$=\frac{(R(\tau)+\frac{1}{3})}{2\xi(\tau)} + \frac{R_r(\tau)}{(R(\tau)+\frac{1}{3})} - \frac{1}{\xi(\tau)} + \frac{1}{\xi(\tau)}-$$
$$-\frac{(R(\tau)+\frac{1}{3})}{2\xi(\tau)} - \frac{(\tau+c)R_\tau(\tau)}{2\xi(\tau)} = \frac{R_\xi(\tau)}{(R(\tau)+\frac{1}{3})} - \frac{(\tau+c)R_\tau(\tau)}{2\xi(\tau)} \text{ but}$$
$$R_\xi(\tau) = \tau_\xi R_\tau(\tau) = \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{2\xi(\tau)}\,R_\tau(\tau)$$
$$\text{then } \frac{d^2\theta(\xi(\tau))}{d\xi^2} = \frac{R_\xi(\tau)}{(R(\tau)+\frac{1}{3})} - \frac{(\tau+c)R_\tau(\tau)}{2\xi(\tau)} = \frac{(\tau+c)(R(\tau)+\frac{1}{3})}{(R(\tau)+\frac{1}{3})2\xi(\tau)}\,R_\tau(\tau)-$$
$$-\frac{(\tau+c)R_\tau(\tau)}{2\xi(\tau)} = 0 \tag{34}$$

3.2. Numerical simulation

Expression (30) is denoted by:

$$R(\xi(\tau)) = 2\sqrt{-\frac{p}{3}}\cos(\frac{\alpha_0}{3})$$
$$\text{where, } p = -\frac{7}{3} + 4\frac{(G+F)}{(\tau+2c)},\quad \alpha_0 = \cos^{-1}(-\frac{q}{2\sqrt{-(\frac{p}{3})^3}}) \text{ and}$$
$$q = -\frac{20}{27} + \frac{4}{3}\frac{(G-2F)}{(\tau+2c)},\ \text{for } Q = (\frac{p}{3})^3 + (\frac{q}{2})^2 < 0,\quad (p<0) \tag{35}$$

where the constants k and c depend on the specific problem's conditions. For example, Eq. (30) must be a straight line only valid for some specific values of c.

The function Q must satisfy the condition $Q < 0$, which may be written as $Q = k$. Now we can find function G, where $k < 0$. Therefore, the function G, once found, is provided as:

$$G = \sqrt[3]{\frac{q_1}{2} - \sqrt{Q_1}} + \sqrt[3]{\frac{q_1}{2} + \sqrt{Q_1}} - (1/3)[(3f_0 - 61/4)/f_1] \text{ where}$$
$$Q_1 = (\frac{p_1}{3})^3 + (\frac{q_1}{2})^2 > 0 \text{ and}$$
$$p_1 = a_1 - \frac{a_2^2}{3},\quad q_1 = 2(\frac{a_2}{3})^3 - \frac{a_1 a_2}{3} + a_0,$$
$$a_0 = \frac{\left[f_0^3 - \frac{61}{4}f_0^2 + f_3 + f_0(64+\frac{9}{2})\right]}{f_1^3},\ a_1 = \frac{\left[3f_0^2 - \frac{61}{2}f_0 + \frac{9}{2}f_2 + 64\right]}{f_1^2},$$
$$a_1 = \frac{\left[3f_0 - \frac{61}{4}\right]}{f_1},\quad f_0 = 3 + \frac{4F}{(\tau+2c)},\quad f_1 = \frac{4}{(\tau+2c)},\quad f_2 = 3 - f_1 F,$$
$$f_3 = \frac{27}{4}f_2^2 - 64f_2 - 27k,\ k<0 \quad \text{and}$$
$$F = \left[\frac{\frac{2(n-3)}{(5-n)^2}\ln(2c+\tau) - (\frac{1-n}{n-5})^{(n+1)}[\ln(2c+\tau)]^n}{n}\right] \text{ for } 1<n<5$$
$$F = \left[\frac{\frac{2(2n-10)}{(9-n)^2}\ln(2c+\tau) - (\frac{n-1}{9-n})^{(n+1)}[\ln(2c+\tau)]^n}{n}\right] \text{ for } 5 \lesssim n < 9$$
$$F = \left[\frac{A\ln(2c+\tau) - B[\ln(2c+\tau)]^n}{n}\right] \quad \text{for other cases} \tag{36}$$

.

Eq. (36) must also be completed according to the condition $Q_1 > 0$, where the arbitrary constant c corresponds to the solution of the Abel's differential Eq. (15). It so happens that function F corresponds to the non-homogeneous part of Eq. (15).

The applicability of Eq. (30) is a fundamental part of the solution to Equation(7), which happens to be obtained from Eq. (15).

3.2.1. Proof of equation (29)

From Eq. (8), and only considering solution $\theta(\xi) = \tau\xi^{\frac{2}{n-5}}$ and substituing it in (1), we obtain:

$$\xi^2\frac{d^2\tau(\xi)}{d\xi^2} + 2\frac{n-3}{n-5}\xi\frac{d\tau(\xi)}{d\xi} + 2\frac{n-3}{(1-n)^2}\tau(\xi) = -\tau^n(\xi)\xi^{\frac{4(n-3)}{n-5}} \tag{37}$$

now, let $\xi = \exp(t)$, then it is obtained:

$$\frac{d^2\tau(t)}{dt^2} + \frac{n-1}{n-5}\frac{d\tau(t)}{dt} + 2\frac{n-3}{(1-n)^2}\tau(t) = -\tau^n(t)\exp(\frac{4(n-3)}{n-5}t) \tag{38}$$

and let $\tau = \tau_3\exp(\beta_1 t)$, substituing in Eq. (38), it is obtained:

$$\frac{d^2\tau_3(t)}{dt^2} + \left[2\beta_1 + \frac{n-1}{n-5}\right]\frac{d\tau_3(t)}{dt} + \left[\beta_1^2 + \frac{n-1}{n-5}\beta_1 + 2\frac{n-3}{(1-n)^2}\right]\tau_3(t)=$$
$$= -\tau_3^n(t)\exp(\beta_1 n - \beta_1 + \frac{4(n-3)}{n-5}) \tag{39}$$

then, $\beta_1 n - \beta_1 + \frac{4(n-3)}{n-5} = 0 \Longrightarrow \beta_1 = -\frac{4(n-3)}{(n-5)(n-1)}$, Eq. (39) changes to:

$$\frac{d^2\tau_3(t)}{dt^2} + \frac{n-5}{n-1}\frac{d\tau_3(t)}{dt} + 2\frac{3-n}{(n-5)^2}\tau_3(t) = -\tau_3^n(t) \to -\frac{\tau_3^n(t)}{n} \tag{40}$$

which is Eq. (24) and its solution is Eq. (26). Therefore $\tau = \tau_3$ and $\xi_1 = \xi^{\frac{n-1}{5-n}}$, and consequently $\Theta(\xi) = \tau\xi_1^{\frac{2}{1-n}} = \tau\xi^{\frac{2}{n-5}}$.

3.3. Case II for $5 \lesssim n < 9$

The solution $\theta(\xi) = \tau(\xi)\xi^{\frac{2}{1-n}}$ is substituted into (1) to obtain: $\xi^2\frac{d^2\tau(\xi)}{d\xi^2} + 2\frac{3-n}{1-n}\xi\frac{d\tau(\xi)}{d\xi} + 2\frac{3-n}{(n-1)^2}\tau(\xi) = -\xi^2\tau(\xi)^n$, and substituing $\xi = \exp(t)$, it is obtained $\frac{d^2\tau(t)}{dt^2} + \frac{5-n}{1-n}\xi\frac{d\tau(t)}{dt} + 2\frac{3-n}{(n-1)^2}\tau(t) = -\exp(2t)\tau(t)^n$, now substituing in last equation $\tau(t) = \tau_0(t)\exp(\alpha t)$, then it is obtained that $\alpha = \frac{2}{1-n}$ and

$$\frac{d^2\tau_0(t)}{dt^2} + \frac{9-n}{1-n}\xi\frac{d\tau_0(t)}{dt} + 2\frac{10-2n}{(n-1)^2}\tau_0(t) = -\tau_0(t)^n \to \frac{\tau_0^n(t)}{n} \tag{41}$$

By following the same procedure used to obtain from Eqs. (25)–(29), which is how we obtained $\theta(\xi) = \tau\xi^{\frac{4}{n-9}}$ (assuming $\tau = \tau_0$), and substituting it in (1), we obtain:

$$\xi^2\frac{d^2\tau(\xi)}{d\xi^2} + \left[\frac{8}{n-9} + 2\right]\xi\frac{d\tau(\xi)}{d\xi} + \frac{4}{n-9}\left(\frac{4}{n-9} - 1\right)\tau(\xi)$$
$$= -\xi^{2+\frac{4}{n-9}(n-1)}\tau^n \tag{42}$$

now, let $\xi = \exp(t)$ and $\tau = \tau_4\exp(\beta_2 t)$, we obtain:

$$\frac{d^2\tau_4(t)}{dt^2} + \left[2\beta_2 + \frac{n-1}{n-9}\right]\frac{d\tau_4(t)}{dt} + \left[\beta_2^2 + \frac{4}{n-9}\beta_2 + 4\frac{n-5}{(n-9)^2}\right]\tau_4(t)=$$
$$= -\tau_4^n(t)\exp(\beta_2 n - \beta_2 + \frac{6n-22)}{n-9}) \tag{43}$$

when $\beta_2 = -\frac{6n-22}{(n-1)(n-9)}$, it is obtained:

$$\frac{d^2\tau_4(t)}{dt^2} + \left[\frac{n-5}{n-1}\right]\frac{d\tau_4(t)}{dt} + \left[\frac{3-n}{(n-1)^2}\right]\tau_4(t) = -\tau_4^n(t) \to -\frac{\tau_4^n(t)}{n} \tag{44}$$

which happens to be identical to Eq. (24).

3.3.1. Solution behavior for $\theta(\xi(\tau))$ and $\frac{d\theta(\xi(\tau))}{d\xi}$

In order to obtain coefficients a, b and d, given by Eqs. (8) and (9), we applied the initial conditions given by Eq. (22). Such coefficients are obtained through Eqs. (45) and (46), as shown numerically in tables 1 and 2.

$$\begin{aligned} d &= \frac{-1}{((\frac{1}{\xi}|_{\xi=\xi_0} - (\frac{1}{\xi}|_{\xi=0})\tau_\xi|_{\xi=0} + \xi^{\beta}\tau|_{\xi=\xi_0})} \\ b &= d\tau_\xi \ and \ a = 1 - b \text{ where } \beta = \frac{2}{n-5} \text{ for the interval } 1 < n < 5 \\ and \ \beta &= \frac{4}{n-9} \ for \ the \ interval \ 5 \lesssim n < 9 \end{aligned} \tag{45}$$

and the parameters *a, b, d* of the solutions are subject to the boundary conditions given by Eq. (23), then it is obtained **;**

$$\begin{aligned} d &= \frac{-1}{(\tau_{\xi\xi}|_{\xi=0} + 2\frac{3-n}{5-n}\tau_\xi|_{\xi=0})} \\ b &= d\tau_\xi \ and \ a = 1 - b \ for \ the \ interval \ 1 < n < 5 \\ d &= \frac{-1}{(\tau_{\xi\xi}|_{\xi=0} + 2\frac{7-n}{9-n}\tau_\xi|_{\xi=0})} \\ b &= d\tau_\xi \ and \ a = 1 - b \ \ for \ the \ interval \ 5 \lesssim n < 9 \end{aligned} \tag{46}$$

where $\tau_{\xi\xi} = \frac{d^2\tau}{d\xi^2}$ and $\tau_\xi = \frac{d\tau}{d\xi}$ are defined in Eqs. (32) and (35). Note that the analytical solutions agree well with both boundary conditions, see Figs. 2 to 7 and tables 1 and 2.

n	c	k	d	b	a
1.5	1.15	− 0.65	− 0.484	− 1.144	2.144
2	1.2	− 0.7	− 0.5746	− 1.4349	2.4349
2.592	1.4	− 0.75	− 0.9315	− 2.7436	3.7436
3	1.61	− 0.9	4.564	15.3187	− 14.3187
3.23	0.865	− 0.86	1.6404	3.1412	− 2.1412

(47)

Table 1.-The valued parameters for $1 < n < 5$,
. and

n	c	k	d	b	a
5	0.876	− . 006115	1.4611	1.6505	− 0.6505

(48)

Table 2.- The valued parameters for $5 \lesssim n < 9$

We next prepared five cases for analysis with regards to the influence of n on solutions of Eq. (1); for real stars, we have $n = 1.5$, $n = 2$, $n = 3$, and for exoplanet systems $n = 2.592$ and $n = 3.23$, for the interval $1 < n < 5$.

Figures 2 to 6 show the proposed solution $\theta(\xi(\tau))$ given in Eq. (8), the exact polytrope function. It is proven that boundary conditions (*b.c.*) of Eqs. (22) and (23) are satisfied for $n = 1.5$, 2, 2.592,3 and 3.23. Also, the behavior of $\frac{d\theta(\xi(\tau))}{d\xi}$ is shown.

Fig. 7 shows that for $n \simeq 5$, (for interval $5 \lesssim n < 9$), the proposed solution satisfies Eq. (9), with the boundary conditions given by (22) and (23), for $\frac{d\theta(\xi(\tau))}{d\xi}$ and $\theta(\xi(\tau))$.

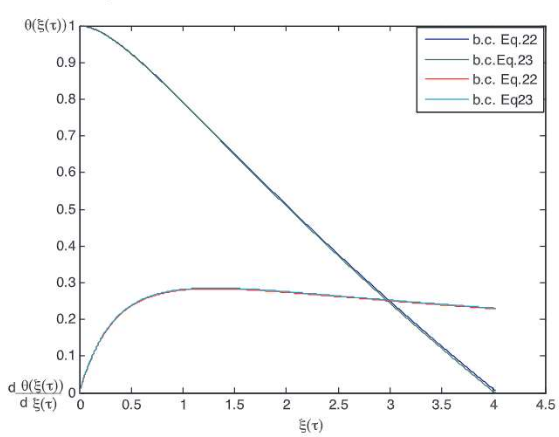

Fig. 2. For $n = 1.5$.

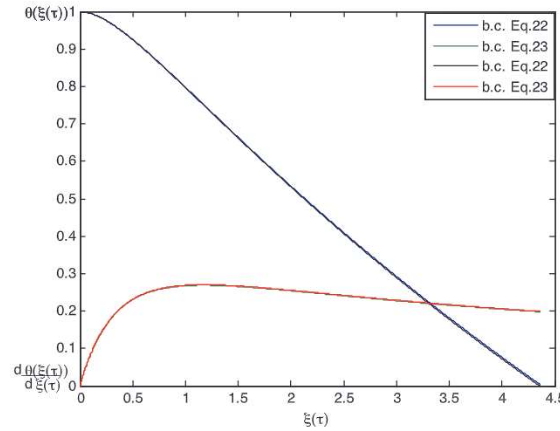

Fig. 3. For $n = 2$.

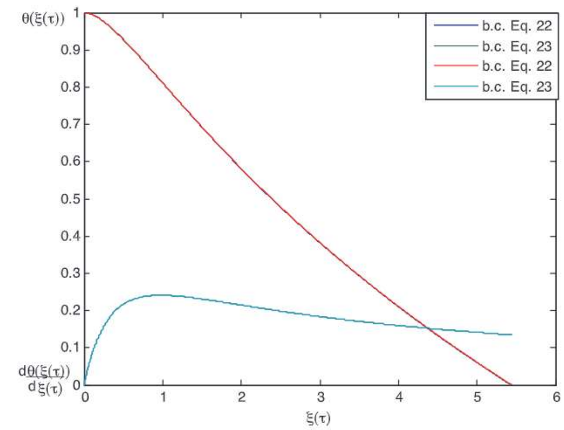

Fig. 4. For $n = 2.592$.

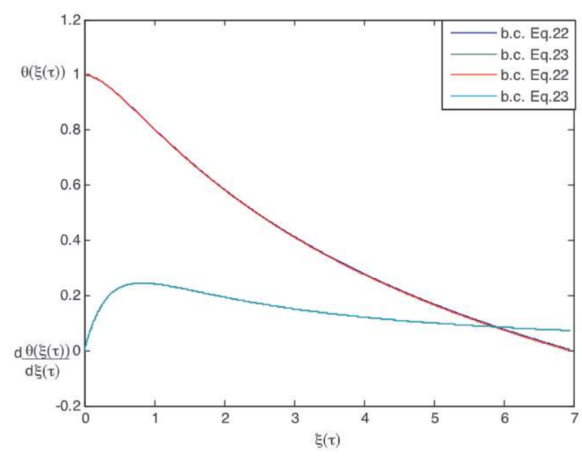

Fig. 5. For $n = 3$.

The values used in Eq. (45) of ξ_0, which is the first root of $\theta(\xi)$ are: for $n = 3$, $\xi_0 = 6.8965$; for $n = 1.5$, $\xi_0 = 3.65375$; for $n = 2$, $\xi_0 = 4.35$ (see edd (1926)); for $n = 2.592$, $\xi_0 = 5.58$; for $n = 3.23$, $\xi_0 = 7.91$, where $\theta(\xi)|_{\xi_0} = 0$. Figures 2 through 6 show that the sphere's radius is finite and borders its surface. Fig. 7, on the other hand, shows the behavior of $\theta(\xi)|_{\xi_0\to\infty}$, where the sphere's radius extends infinitely for $\xi_0 \to \infty$.

Figures 2 to 6 apply particularly for $n = 3.23$, 3, 2.592, 2 and 1.5, as used in Eq. (8) and the approximation of c and k given by Table 1. And, considering the case $n \simeq 5$ as used Eq. (9), and the approximation of selected values of the parameters c and k as shown at Table 2.

We show the influence of n on the solutions given by Eq. (1). We have studied many of the mathematical properties of the boundary

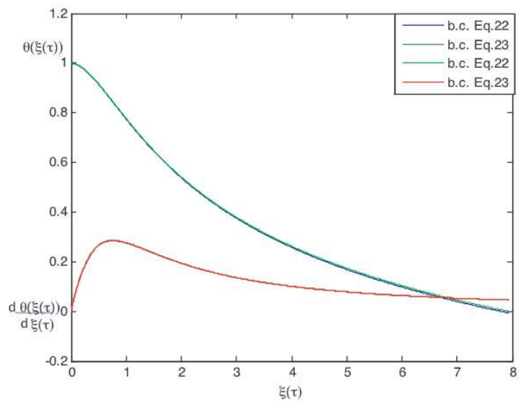

Fig. 6. For $n = 3.23$.

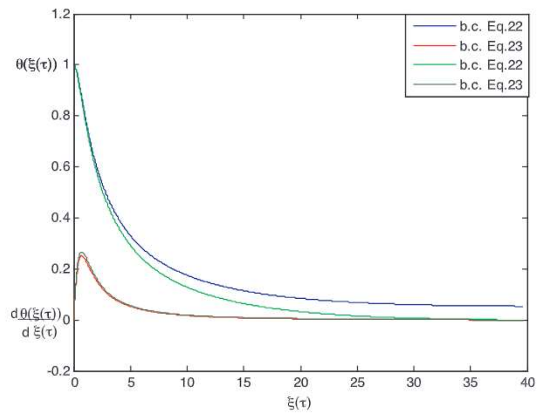

Fig. 7. For $n \simeq 5$, also the boundary condition Eq. (23) is not complete for $\theta(\xi)$ due to ξ_0 is not a root of $\theta(\xi)$.

conditions given by Eqs. (22) and (23). Now, we turn our attention to determining the influence of the coefficients c and k. These parameters determine the solution profiles, given by Eqs. (30) and (36).

Fig. 8 shows the comparison between the different solution of Eq. (8).

Now, Fig. 9 shows the case $n \simeq 5$ used in Eq. (1) and its solution given by Eq. (9), and the approximation of selected values of the parameters c and k, as shown at Table 2.

Fig. 10 shows the comparison between the solution of Eq. (8) and the numerical solution (dots curve) obtained by edd (1926).

Figs. 11 and 12 show the Lane–Emden equation of the first kind for $\theta(\xi(\tau))$ and $\frac{d\theta(\xi(\tau))}{d\xi(\tau)}$, for $n = 2$ and 3, which is compared graphically with

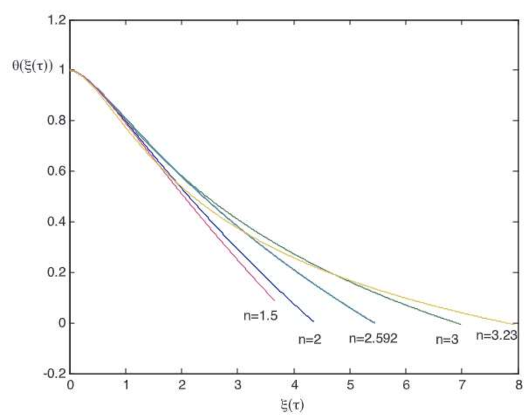

Fig. 8. The proposed solution for Eq. (8) that satisfies all boundary conditions of Eq. (22) is shown.

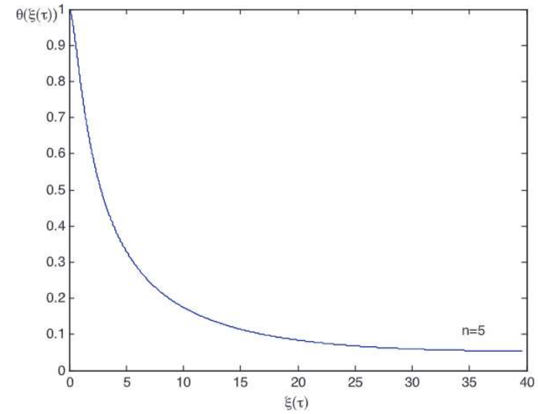

Fig. 9. The proposed solution for Eq. (9), which satisfies all boundary conditions of Eq. (22).

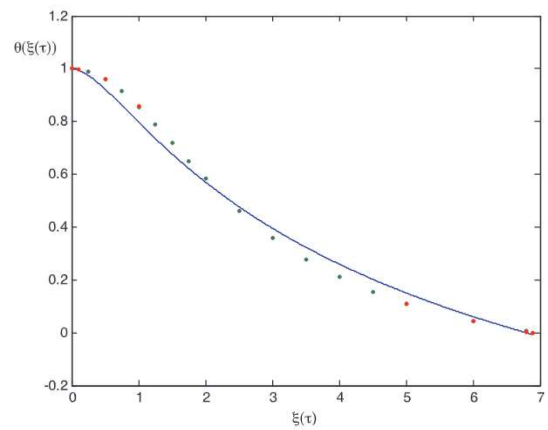

Fig. 10. The Lane–Emden equation of the first kind for $\theta(\xi)$ when $n = 3$ is compared graphically with the deriving of the analytical Eq. (8) and the numerical solution (as shown by the dot curve).Fig. 10. The Lane–Emden equation of the first kind for $\theta(\xi)$ when $n = 3$ is compared graphically with the deriving of the analytical Eq. (8) and the numerical solution (as shown by the dot curve).

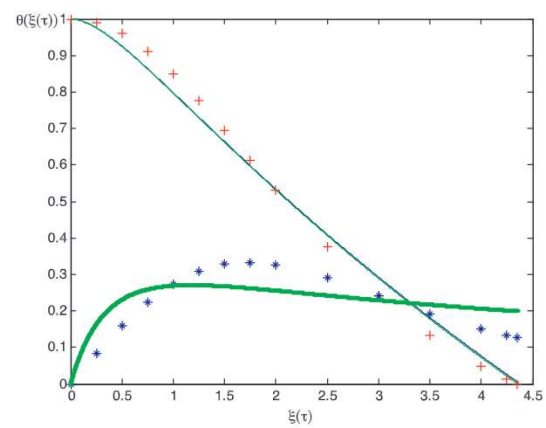

Fig. 11. The proposed solution for Eq. (8) satisfies all boundary conditions of Eq. (22), for $n = 2$.

the deriving of the analytical Eq. (8) and the numerical solutions obtained by edd (1926), as plotted:

Fig. 13 shows the solutions for the Lane–Emden equation of the first kind for $\theta(\xi(\tau))$, $\frac{d\theta(\xi(\tau))}{d\xi(\tau)}$ and $\theta(\xi)$, $\frac{d\theta(\xi)}{d\xi}$ for $n \simeq 5$. They are then graphically compared with the analytical solution given by (9), the numerical solutions given by edd (1926), and the exact solution $\Theta(\xi) = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$.

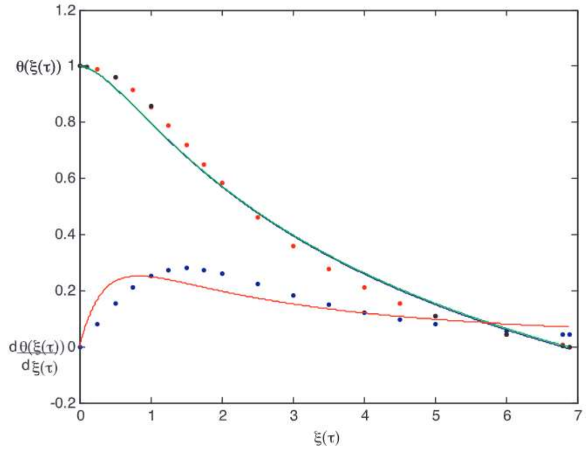

Fig. 12. The proposed solution for Eq. (8) satisfies all boundary conditions of Eq. (22). The case $n = 3$ corresponds to an adiabatic star supported by the pressure of ultra-relativistic gas.

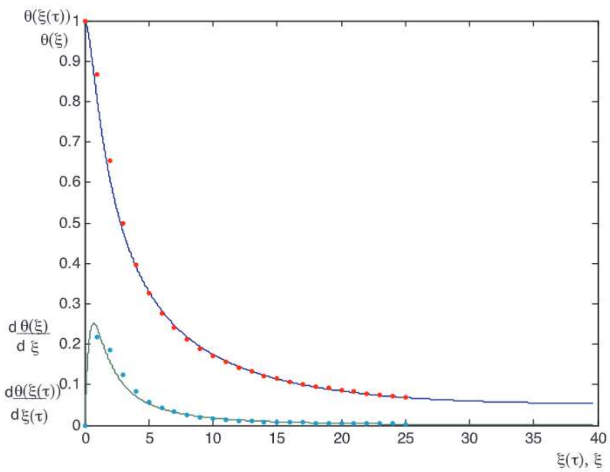

Fig. 13. The proposed solution for Eq. (9) satisfies all boundary conditions of Eq. (23).

In fact, the results are almost indistinguishable from the solution given by Eq. (9) and the exact solution $\theta(\xi)= \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$.

4. Conclusions

We have presented the exact and analytical solution to the nonlinear singular Lane-Emden equation of the first kind. The Lane–Emden type equation describes numerous problems in mathematical physics (astrophysics, in particular). In astrophysics, the equation of the first kind describes the equilibrium of non-rotating polytropic fluids in a self-gravitating star. Through the temperature variation subjected to the laws of classical thermodynamics, the spherical symmetry and the gas cloud are affected by molecular attractions.

In many reported studies, the function $\theta_g(\xi) = \frac{1}{\xi}$ is a homogeneous solution of the corresponding nonhomogeneous Equation (1). Though, it is usually not taken into account because the limit $\theta_g(\xi \to 0) \to \infty$, and because it does not satisfy the boundary condition $\theta(\xi \to 0) \to$ finite value. In our analytical solution, as introduced by Eqs. (8) and (9), we showed that the function $\theta_g(\xi) = \frac{1}{\xi}|_{\xi=0} \to$finite value is satisfied, Likewise the boundary condition, $\theta(\xi \to 0) = \frac{1}{\xi} = 1$ is satisfied too.

Now, considering the case of the degenerate gas cloud for $n = 1.5$ (only in this case, our proposed solution for the first root is obtained as $\xi_0 \to 4$ and not $\xi_0 \to 3.65$, as reported in the literature) and $n = 3$, substituting these values in Eq. (8), the boundary conditions are complete. The solutions given by Equations (8) and (9) are validated for the intervals, $1 < n < 5$ and $5 \lesssim n < 9$, excluding $n = 0, 1, 5$. The solution given by $\theta(\xi)$ allows us to study the energy transport through mass transfer between the star's different levels, defined by $\frac{d\theta(\xi)}{d\xi}$ and mass change rate$\frac{d^2\theta(\xi)}{d\xi^2}$.

By comparing the parameters *a, b* and *d* from tables 1 and 2, it can be validated that for $n = 3$ and $n \simeq 5$, the arithmetic signs are changed. Moreover, to determine *c* and *k*, they must be ajusted in such a way that $\theta(\xi(\tau)) = \xi(\tau)$ must behave as a straight line.

The nonlinear Lane–Emden equation has been solved in this work. Another main advantage for the solution is the analytic expression for the mass $M(r)$, and the polytropic temperature $\theta(\xi)$. Lastly, the proposed solution applies, with really good precision, to the modeling of the structures of stars and galaxies.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.newast.2020.101458

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