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Exceptional Algebraic Sets for Infinite Discrete Groups of $PSL(3, \mathbb{C})$

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Abstract
In this note we show that the exceptional algebraic set for an infinite discrete group in $PSL(3, \mathbb{C})$ should be a finite union of: complex lines, copies of the Veronese curve or copies of the cubic $xy^2 - z^3$.

Keywords Complex Kleinian groups · Exceptional algebraic set · Veronese groups.

Mathematics Subject Classification Primary 37F99; Secondary 30F40 · 20H10 · 57M60

Introduction

Complex Kleinian groups first appeared in mathematics with the works of Henri Poincaré, as a way to qualitatively study the solutions of ordinary differential equations of order two, one can say that the success of Poincaré was because he managed to establish a dictionary between differential equations and group actions. Subsequently, this theory achieved a new boom with the introduction of quasi-conformal maps and the discovering of bridges between hyperbolic three-manifolds this theory. At the beginning of the '90s Verjovsky and Seade began studying, see Seade and Verjovsky (2002), the discrete groups of projective transformations that act in projective spaces, as a proposal to establish a dictionary between actions of discrete groups and the theory of foliations, partial and ordinary differential equations. The purpose of this

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note is to understand those infinite discrete groups whose dynamic could be described in terms of a group acting in an algebraic curve in the complex-projective plane, i. e. Kleinian groups leaving an invariant algebraic curve, analogous results but in the case of iteration holomorphic maps in $\mathbb{P}^n_{\mathbb{C}}$ have been studied extensively, see (Briend et al. 2004; Cerveau and Lins 2000; Fornaes and Sibony 1994).

Recall that in the classical case, see (Greenberg 1962), a set in $\mathbb{P}^1_{\mathbb{C}}$ is said to be exceptional for the action of an infinite group $G \subset \mathrm{PSL}(2,\mathbb{C})$, discrete or not, if it is invariant under the action of G and is a finite set. In analogy with this, we will say that $S \subset P^2$ is an *exceptional algebraic set* for the action of an infinite group, in our case discrete, if it is G-invariant and a complex algebraic curve, compare with the definition of algebraically mixing in Seade and Verjovsky (2002). As we will see, the geometry of the exceptional algebraic sets is very restricted and in consequence, the class of groups with an exceptional set is small, more precisely in this article, we show:

Theorem 0.1 *Let $G \subset PSL(3,\mathbb{C})$ be an infinite discrete group and S a complex algebraic surface invariant under G, if S_0 is an irreducible component of S, then S_0 is a complex line, the Veronese curve or the projective curve induced by the polynomial $p(x,y,z) = xy^2 - z^3$.*

Theorem 0.2 *Let $G \subset PSL(3,\mathbb{C})$ be an infinite discrete group. If G is a non-virtually commutative group and S is a complex algebraic surface invariant under G, then S is either a finite union of lines or the Veronese curve. Moreover, if S has at least four lines, then S contains at most three non-concurrent lines.*

Theorem 0.3 *Let $G \subset PSL(3,\mathbb{C})$ be infinite discrete groups. If G is virtually cyclic group and S a complex algebraic surface invariant under G, then S is:*

1. *A finite union of lines. Moreover, in such a collection of lines the largest number of non-concurrent lines is three.*
2. *A finite union of copies of the Veronese curve and a finite union (possibly empty) of lines which are either tangent and secant to the copies of the Veronese curve. Moreover the number of lines which are tangents does not exceed two and the number of secants is at most one.*
3. *A finite union of copies of the cubic induced by the polynomial $xy^2 - z^3$ and finite union (possibly empty) on tangent and secant lines. Moreover the number of tangents does not exceed two and the number of secants is at most one.*

Corollary 0.4 *Let $G \subset PSL(3,\mathbb{C})$ be a discrete group with an algebraic exceptional set, then G is either virtually affine or is the representation of a Kleinian group of Möbius transformations though the irreducible representation of $Mob(\hat{\mathbb{C}})$ into $PSL(3,\mathbb{C})$, see example 2.2 below.*

The paper is organized as follows: Sect. 1 reviews some elementary well-known facts on two-dimensional complex Kleinain groups, see (Cano et al. 2013) for extra information. In Sect. 2, we provide a collection of examples that depicts all the possible ways to constructing groups with exceptional algebraic surfaces as well as the respective surfaces. In Sect. 3 we show that every irreducible component of an exceptional algebraic set is a line, a copy of the Veronese curve of a cubic with a cusp as a singularity, the proof of this fact relies strongly in the use of the Plücker Formulas as well as in the Hurwitz theorems for curves, most of the material on algebraic curves used in this note is well known but the interested reader could see (Fischer 2001; Miranda 1995; Shafaverich 1994) for full details. Finally in Sect. 4 we prove the main theorem of this article.

1 Preliminaries

In this section we establish some elementary facts that we use in the sequel.

1.1 Projective Geometry

The complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is the quotient space $(\mathbb{C}^3 - \{\mathbf{0}\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^3 - \{\mathbf{0}\}$ by the usual scalar multiplication. Let $[\ ] : \mathbb{C}^3 - \{\mathbf{0}\} \to \mathbb{P}^2_{\mathbb{C}}$ be the quotient map. A set $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ is said to be a complex line if $[\ell]^{-1} \cup \{\mathbf{0}\}$ is a complex linear subspace of dimension 2. Given $p, q \in \mathbb{P}^2_{\mathbb{C}}$ distinct points, there exists a unique complex line passing through p and q, such line is denoted by $\overleftrightarrow{p, q}$. The set of all complex lines in $\mathbb{P}^2_{\mathbb{C}}$, denoted $Gr(\mathbb{P}^2_{\mathbb{C}})$, equipped with the topology of the Hausdorff convergence, actually is diffeomorphic to $\mathbb{P}^2_{\mathbb{C}}$ and it is its projective dual $\mathbb{P}^{*2}_{\mathbb{C}} \cong Gr(\mathbb{P}^2_{\mathbb{C}})$.

Consider the action of $\mathbb{Z}_3$ (viewed as the cubic roots of the unity) on $\mathrm{SL}(3, \mathbb{C})$ given by the usual scalar multiplication. Then

$$\mathrm{PSL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C})/\mathbb{Z}_3\,,$$

is a Lie group whose elements are called projective transformations. Let $[[\ ]] : \mathrm{SL}(3, \mathbb{C}) \to \mathrm{PSL}(3, \mathbb{C})$ be the quotient map, $g \in \mathrm{PSL}(3, \mathbb{C})$ and $\mathbf{g} \in \mathrm{GL}\,(3, \mathbb{C})$. We say that $\mathbf{g}$ is a lift of g if there exists a cubic root α of $Det(\mathbf{g})$ such that $[\alpha^{-1}\mathbf{g}] = g$, by abuse of notation in the following we will use $[\ ]$ instead $[[\ ]]$. We use the notation (g_{ij}) to denote elements in $\mathrm{SL}(3, \mathbb{C})$. It is easy to show that $\mathrm{PSL}(3, \mathbb{C})$ acts transitively, effectively and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ by $[\mathbf{g}]([w]) = [\mathbf{g}(w)]$, where $w \in \mathbb{C}^3 - \{\mathbf{0}\}$ and $\mathbf{g} \in \mathrm{GL}\,(3, \mathbb{C})$.

If g is an element in $\mathrm{PSL}(3, \mathbb{C})$ and $\mathbf{g} \in \mathrm{SL}(3, \mathbb{C})$ is a lift of g, we say , see (Cano et al. 2013), that:

- g is a elliptic if $\mathbf{g}$ is diagonalizable with unitary eigenvalues.
- g is parabolic if $\mathbf{g}$ is non-diagonalizable with unitary eigenvalues.
- g is loxodromic if $\mathbf{g}$ has some non-unitary eigenvalue.

Clearly this definition does not depend on the choice of the lift.

Lemma 1.1 (See Lemma 6.6 in Cano and Seade 2014)
Let G be an infinite discrete group, then G contains an element g with infinite order. Moreover, g is either parabolic or loxodromic.

It is possible to decide if the element in the previous lemma is parabolic or loxodromic but one should impose dynamic restrictions over G, a detailed discussion of this conditions goes beyond the scope of this paper, the interested reader see (Barrera et al. 2020).

Let $M(3,\mathbb{C})$ be the set of all 3×3 matrices with complex coefficients. Define the space of pseudo-projective maps by: $\mathrm{SP}(3,\mathbb{C}) = (M(3,\mathbb{C})-\{\mathbf{0}\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $M(3,\mathbb{C})-\{\mathbf{0}\}$ by the usual scalar multiplication. We have the quotient map $[\,]: M(3,\mathbb{C})-\{\mathbf{0}\}\to \mathrm{SP}(3,\mathbb{C})$. Given $P\in \mathrm{SP}(3,\mathbb{C})$ we define its kernel by:

$$Ker(P) = [Ker(\mathbf{P})-\{\mathbf{0}\}]\,,$$

where $\mathbf{P}\in M(3,\mathbb{C})$ is a lift of P. Clearly $\mathrm{PSL}(3,\mathbb{C})\subset \mathrm{SP}(3,\mathbb{C})$ and an element P in $\mathrm{SP}(3,\mathbb{C})$ is in $\mathrm{PSL}(3,\mathbb{C})$ if and only if $Ker(P)=\emptyset$. Notice that $\mathrm{SP}(3,\mathbb{C})$ is a manifold naturally diffeomorphic to $\mathbb{P}^8_{\mathbb{C}}$, so it is compact. For each $P\in \mathrm{SP}(3,\mathbb{C})$ with lift $p\in M(3,\mathbb{C})-\{\mathbf{0}\}$ we can define an holomorphic funtion $P:\mathbb{P}^2_{\mathbb{C}}-Ker(P)\to \mathbb{P}^2_{\mathbb{C}}$ by $P[w]=[p(w)]$. The following lemma relates the convergence of projective maps as pseudo-projective maps and the converge holomorphic functions, also can be considered as a generalization of the convergence property of Möbius transformations, see page 44 in Kapovich (2010).

Lemma 1.2 (See Proposition 1.1 in Cano et al. 2017) *Let $(g_n)\subset PSL(3,\mathbb{C})$ be a sequence of projective transformations, then there is a subsequence (h_n) of (g_n) and $h\in SP(3,\mathbb{C})$ such that:*

1. *The sequence h_n converges to h as elements in $SP(3,\mathbb{C})$.*
2. *Considering h_n and h as holomorphic functions from $\mathbb{P}^2_{\mathbb{C}}-Ker(h)$ to $\mathbb{P}^2_{\mathbb{C}}$ we have h_n converges to h uniformly on compact sets of $\mathbb{P}^2_{\mathbb{C}}-Ker(h)$.*

The following proposition is essentially due in Navarrete (2008).

Proposition 1.3 (See Lemmas 5.5 and 6.10 in Navarrete 2008)

1. $\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}^n$ *converges to* $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$.
2. $\begin{bmatrix}\alpha&1&0\\0&\alpha&0\\0&0&\alpha^{-2}\end{bmatrix}^n$ *converges to* $\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$, *if* $|\alpha|\geq 1$.
3. $\begin{bmatrix}\alpha&1&0\\0&\alpha&0\\0&0&\alpha^{-2}\end{bmatrix}^n$ *converges to* $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, *if* $|\alpha|<1$.
4. $\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}^n$ *converges to* $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, *if* $\alpha\beta\gamma=1,\ |\alpha|<|\beta|<|\gamma|$.
5. $\begin{bmatrix}\alpha&0&0\\0&e^{2\pi i\theta}\alpha&0\\0&0&\alpha^{-2}e^{-2\pi i\theta}\end{bmatrix}^n$ *converges to* $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$, *if* $|\alpha|<1$.

A subgroup $G \subset PSL(3, \mathbb{C})$ is weakly semi-controllable, if it have a global fixed point $p \in \mathbb{P}^2_{\mathbb{C}}$; hence for each line $\mathcal{L}$ in $\mathbb{P}^2_{\mathbb{C}} - \{p\}$ one has a canonical holomorphic projection map π from $\mathbb{P}^2_{\mathbb{C}} - \{p\}$ into $\mathcal{L} \cong \mathbb{P}^1_{\mathbb{C}}$. This defines a group morphism:

$$\begin{gathered} \Pi = \Pi_{p,\ell,G} : G \to Bihol(\ell) \cong \mathrm{PSL}(2, \mathbb{C}) \\ \Pi(g)(x) = \pi(g(x)) \end{gathered}$$

which essentially is independent of all choices, see page 133 in Cano et al. (2013) for details.

2 Examples

In this section we present examples of curves invariant under Lie groups later we will see that any irreducible curve is one of the curves presented here.

Example 2.1 (Lines) The group of affine projective transformations.

Example 2.2 (Veronese Curves) Recall the Veronese embedding is given by

$$\begin{aligned} &\psi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \\ &\psi([z, w]) = \left[z^2, 2zw, w^2\right]. \end{aligned}$$

and the unique irreducible representation of $PSL(2, \mathbb{C})$ into $PSL(3, \mathbb{C})$ is given ι : $\mathrm{PSL}(2, \mathbb{C}) \to \mathrm{PSL}(3, \mathbb{C})$ where

$$\iota\left(\frac{az+b}{cz+d}\right) = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & dc & d^2 \end{bmatrix}.$$

Then $\iota PSL(2, \mathbb{C})$ leaves invariant $Ver = \psi(\mathbb{P}^1_{\mathbb{C}})$.

Example 2.3 (Cubic with a cusp) Consider the homogeneous cubic polynomial $p(x, y, z) = xy^2 - z^3$ then the projective curve induced by p is a cubic with a cusp in [1:0:0] and a inflection point in [0:1:0]. Moreover, the cubic and the lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_3, e_2}$ and $\overleftrightarrow{e_1, e_3}$ are invariant under the Lie group

$$\mathbb{C}^*_p = \left\{ \begin{pmatrix} a^{-5} & 0 & 0 \\ 0 & a^4 & 0 \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{C}^* \right\}$$

Example 2.4 (Pencil of lines) Consider the Lie group given by

$$\mathbb{C}^2_\infty = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Then $\mathbb{C}^2_\infty$ leaves invariant any line passing trough $[1 : 0 : 0]$.

3 Geometry and dynamic of the invariant curves

Lemma 3.1 *Let $G \subset PSL(3, \mathbb{C})$ be a discrete infinite group and S an algebraic reducible complex curve invariant under G. If S is not a line and $g \in G$ has infinite order then S contains a fixed point of g and the action of g restricted to S has infinite order. Moreover, if S is non-singular, then S has genus 0.*

Proof By Lemma 1.1 there is an element $g \in G$ with infinite order which is either loxodromic or parabolic. Consider the following cases:
Case 1: g is loxodromic. By Proposition 1.3 there is a point p an a line ℓ such that: $p \notin \ell$, $p \cup \ell$ is invariant under g and $(g^n)_{n \in \mathbb{N}}$ converges uniformly on compact sets of $\mathbb{P}^2_{\mathbb{C}} - \ell$ to p.

Claim 1: $p \in S$. Since S is not a line by Bézout Theorem we know that $S \cap \ell$ is a finite set, thus we can find a point $q \in S - \ell$. Since S is closed and invariant under g, we conclude $p \in S$.

Case 2: g is parabolic. By Proposition 1.3 there is a point p an a line ℓ such that: $p \in \ell$, p, and ℓ are invariant under g and $(g^n)_{n \in \mathbb{N}}$ converges uniformly on compact sets of $\mathbb{P}^2_{\mathbb{C}} - \ell$ to p. As in the previous case we can show that $p \in S$.

Finally observe that for any point q in $S - \ell$ the set $\{g^n q\}$ is infinite, which shows that the action of g restricted to S has infinite order.
Now let us assume that S is non-singular. If $\pi_1(S)$ is non-trivial, then there exists a non-trivial class in $\pi_1(S)$, say h. Let us assume without lost of generality that G contains a loxodromic element g, the proof in the parabolic case will be similar. Then there is a point p an a line ℓ such that: $p \notin \ell$, $p \cup \ell$ is invariant under g and $(g^n)_{n \in \mathbb{N}}$ converges uniformly on compact sets of $\mathbb{P}^2_{\mathbb{C}} - \ell$ to p. Recall that h can be assumed as the homotopy class of a path $\tilde{h}$ based in p, since $S \cap \ell$ is finite we can assume that $\tilde{h}$ is a loop based on p that does not contains points in ℓ. For N large we have that $g^N(\tilde{h})$ is contained in a simply connected neighborhood of p that is $g^N_{\#} h$ is trivial, thus $g^N_{\#} : \pi_1(S) \to \pi_1(S)$ is not a group isomorphism, which is a contradiction, since g^N is a homeomorphism. □

As an immediate consequence of the proof of the previous lemma we have:

Corollary 3.2 *If $M \subset \mathbb{P}^2_{\mathbb{C}}$ an embedded k-manifold with $k \geq 2$ invariant under a discrete group $G \subset PSL(3, \mathbb{C})$, then S is simply connected*

Lemma 3.3 *Let $G \subset PSL(3, \mathbb{C})$ be a discrete group and S a G-invariant complex irreducible algebraic curve. Then the dual group G^* leaves invariant the dual algebraic curve S^*. Moreover, if S has singularities then S^* does.*

Proof Let $p \in S^*$ then there is a sequence of points $(p_n) \in S^*$ such that each p_n converges to p and for each n we have p_n^* is a tangent line to S at a non-singular point. Since the action of G is by biholomorphism of $\mathbb{P}^2_{\mathbb{C}}$ we deduce for each $\gamma \in \Gamma$ we have $\gamma(p_n^*)$ is a tangent line to S at a non-singular point. Trivially $\gamma(p_n^*)$ converges to $\gamma(p^*)$.

Let us prove the other part of the Lemma, let us assume that S^* is non-singular, then by the Riemann-Hurwitz formula we have

$$0 = (n-1)(n-2)$$

where n is the degree of S^*. Thus S^* is a line or S is quadratic, since $S^{**} = S$, see page 74 in Fischer (2001), we deduce that S^* is quadratic. By the Plücker class formula, we have the degree of S is given by $n(n-1) = 2$. Since every quadratic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to the Veronese curve and the Veronese curve is non-singular we deduce S is non-singular, which is a contradiction. $\square$

Lemma 3.4 *Let $G \subset PSL(3, \mathbb{C})$ be a discrete group and S a G-invariant complex irreducible algebraic curve. If S has singularities then S is a cubic with one node and one inflection point.*

Proof It's well known that S has a finite number of singularities. Since Γ acts on S by biholomorphisms of $\mathbb{P}^2_{\mathbb{C}}$ we conclude that Γ takes singularities of S into singularities of S, thus $\Gamma_0 = \bigcap_{p \in Sing(S)} Isot(\Gamma_0, p)$, here $Sing(S)$ denotes the singular set of S, is a finite index subgroup of Γ.
Claim 1: The genus of S is 0. Let $\widetilde{S}$ be the desingularization of S, then there is a birrational equivalence $f : \widetilde{S} \to S$. Now, let $\gamma_0 \in \Gamma$ be an element with infinite order, then there is $m \in \mathbb{N}$ such that $\gamma_1 = \gamma_0^n \in \Gamma_1$. since f is a birrational equivalence we can construct $\widetilde{\gamma}_1 : \widetilde{S} - f^{-1}Sing(S) \to \widetilde{S} - f^{-1}Sing(S)$ a biholomorphism such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{S} - f^{-1}Sing(S) \arrow[r, "\widetilde{\gamma}_1"] \arrow[d, "f"] & \widetilde{S} - f^{-1}Sing(S) \arrow[d, "f"] \\ S - Sing(S) \arrow[r, "\gamma_1"] & S - Sing(S) \end{tikzcd} \tag{3.1}$$

Since each arrow in the previous diagram is a biholomorphism we deduce that $\widetilde{\gamma}_1$ admits a holomorphic extension to $\widetilde{S}$. Observe that by Lemma 3.1, γ_1 has a fixed point in S and its action on S has infinite order, then by diagram 3.1 we deduce that $\widetilde{\gamma}_1$ has infinite order and at least one fixed point. Recall that a Riemann surface whose group of biholomorphisms is infinite should have genus 1 or 0, see Theorem in 3.9 in Miranda (1995), and in the case of Riemann surfaces of genus 1 the subgroup of biholomorphism sharing a fixed point should be finite, see Proposition 1.12 in Miranda (1995), which concludes the proof of the claim.
Claim 2: The curve S has at most two singularities. Moreover, the singular set is either a node or at most simple cusp. Let $f, \widetilde{S}, \gamma_1, \widetilde{\gamma}_1$ as in claim 1. Thus $\widetilde{\gamma}_1$ fixes each point in $f^{-1}(Sing(S))$. Since $\widetilde{S}$ has genus 0, we deduce $\widetilde{\gamma}_1$ can be conjugated to a Möbius transformation. Because $\widetilde{\gamma}_1$ has infinite order, we conclude $f^{-1}(Sing(S_j))$ contains at most two points. If p is a singular point in S we have $f^{-1}p$ is either one point or two points. If it is one point, p should be a cusp and in the remaining case p is a simple node. Now it is clear that either S has one simple node or at most two cusp.
Claim 3: S has degree 3 and the singular set is either a simple node or a single cusp. Given that $Sing(S)$ contains only cusp and simple nodes and the genus is 0, by applying Clebsch's genus formula, see page 179 in Fischer (2001), to S and we get:

$$0 = genus(S) = (n-1)(n-2) - 2(d+s)$$

where n is the degree, d is the number of nodes and s is the number of cusp in S. In our case the previous equation implies following possibilities:

$$\begin{cases} d = 1 \; s = 0 \\ d = 0 \; s = 1 \\ d = 0 \; s = 2 \end{cases}$$

Substituting this values in the Clebsch's genus formula we get $n = 3$ and also we conclude the case $d = 0$, $s = 2$ is not possible.
On the hand, by Lemma 3.3 the curve S^* is singular and has degree three, thus by Plücker class formula, see page 89 in Fischer (2001), we obtain:

$$3 = deg(S^*) = deg(S)(deg(S) - 1) - 2d - 3s = 6 - 2d - 3s$$

which is only possible when $d = 0$ and $s = 1$, that is the singular set consist of a single cusp. To conclude the proof we need to use Plücker inflection point formula, recall this formula is given by, see page 89 in Fischer (2001): $s^* = 3deg(S)(deg(S)-2)-6d-8s$, where s^* is the number on inflection points in S. □

Lemma 3.5 *Let $G \subset PSL(3, \mathbb{C})$ be a discrete group and S be an irreducible singular curve invariant under G. Then there is a a projective transformation $\gamma \in PSL(3, \mathbb{C})$ such that gS is the curve induce by the polynomial $p(x, y, z) = xy^2 - z^3$ and $\gamma G \gamma^{-1} \subset \mathbb{C}_p^*$, see example 2.3.*

Proof By Lemma 3.4 we have S is a cubic with a cusp, then there is a projective transformation γ such that γS is the Curve induced by the polynomial $p(x, y, z) = xy^2 - z^3$, see cite Shafaverich (1994). Since G acts by biholomorphism of $\mathbb{P}^2_{\mathbb{C}}$ the group gGg^{-1} leaves invariant the singular point and the inflection point of gS. On the other hand, a straightforward computation shows $\overleftrightarrow{e_2, e_3}$ is the tangent line to gS at $[e_2]$ and $\overleftrightarrow{e_1, e_3}$ is the unique tangent line to gS at $[e_1]$, once again since the action of G on $\mathbb{P}^2_{\mathbb{C}}$ is biholomorphisms we conclude that $\overleftrightarrow{e_2, e_3}$ and $\overleftrightarrow{e_1, e_3}$ are G-invariant, thus $[e_3]$ is fixed by gGg^{-1}. Therefore each element in gGg^{-1} has a diagonal lift. Let $g \in gGg^{-1}$ and $(g_{ij}) \in SL(3, \mathbb{C})$ be a lift of g, considering $[1 : 1 : 1] \in gS$ we conclude:

$$g_{11}g_{22}^2 - g_{33}^3 = 0$$

Using the fact $g_{11}g_{22}g_{33} = 1$, we deduce $g_{22} = g_{33}^4$ and $g_{11} = g_{33}^{-5}$, which concludes the proof. □

4 Proof of the Main Theorems

Since S is an algebraic surface, we know S is a finite union of irreducible curves, say $S = \bigcup_{j=1}^n S_j$, then $G_0 = \bigcap Isot(G, s_j)$ is a finite index subgroup of G, leaving invariant each S_j. If S_j has singularities then by Lemmas 3.4 and 3.5 we have that

S_j is projectively equivalent to the curve induced by $xy^2 - z^3$ and the group G_0 is virtually cyclic. If S_j is non- singular by the Riemann Hurwitz theorem and Lemma 3.1 we deduce that S_j is either a line or a copy of the Veronese curve, this shows Theorem 0.1.

In order to prove Theorem 0.2. Observe, the previous argument also shows that if G is non-virtually cyclic, then each connected component of S is either a line or a copy of the Veronese curve. Let us assume that S contains at least two irreducible component, say S_0 and S_1, also let us assume that S_0 is projectively equivalent to the Veronese curve, then $S_0 \cap S_1$ is finite, non-empty and G_0 invariant. On the other hand, since G_0 leaves invariant S_0 and S_0 is biholomorphicaly equivalent to the sphere Lemma 3.1 ensures that G_0 is virtually commutative, which is a contradiction. Thus if S contains a Veronese curve the curve is exactly the Veronese curve, if each reducible component is a line then by proposition 5.15 in Barrera et al. (2016) the largest number of lines in general position in S is 3.

Now the proofs of Theorem 0.3 and Corollary 0.4 are trivial, so we omit it here.□ It would be interesting to understand how this result changes in the higher dimensional setting as well as in the case when we consider smooth manifold or real algebraic manifold as the invariant sets.

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